TS-Reconfiguration of Dominating Sets in circle and circular-arc graphs *

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Abstract

We study the dominating set reconfiguration problem with the token sliding rule. It consists, given a graph G = (V, E) and two dominating sets D_s and D_t of G, in determining if there exists a sequence $S = \langle D_1 := D_s, \ldots, D_\ell := D_t \rangle$ of dominating sets of G such that for any two consecutive dominating sets D_r and D_{r+1} with r < t, $D_{r+1} = D_r \setminus u \cup v$, where $uv \in E$.

In a recent paper, Bonamy et al. [3] studied this problem and raised the following questions: what is the complexity of this problem on circular arc graphs? On circle graphs? In this paper, we answer both questions by proving that the problem is polynomial on circular-arc graphs and PSPACE-complete on circle graphs.

Keywords: reconfiguration, dominating sets, token sliding, circle graphs, circular arc graphs.

1 Introduction

Reconfiguration problems consist, given an instance of a problem, in determining if (and in how many steps) we can transform one of its solutions into another one via a sequence of elementary operations keeping a solution along this sequence. The sequence is called a *reconfiguration sequence*.

Let Π be a problem and \mathcal{I} be an instance of Π . Another way to describe a reconfiguration problem is to define the *reconfiguration graph* $\mathcal{R}_{\mathcal{I}}$, whose vertices are the solutions of the instance \mathcal{I} of Π , and in which two solutions are adjacent if and only if we can transform the first solution into the second in one elementary step. In this paper, we focus on the so-called REACHABILITY problem which, given an instance \mathcal{I} of a problem Π and two solutions I, J of \mathcal{I} , returns true if and only if there exists a reconfiguration sequence from I to J keeping a solution all along. Other works have focused on slightly different problems such as the connectivity of the reconfiguration graph or its diameter, see e.g. [4, 7, 8]. Reconfiguration problems arise in various fields such as combinatorial games, motion of robots, random sampling, or enumeration. Reconfiguration has been intensively studied for various rules and problems such as satisfiability constraints [7], graph coloring [1, 6], vertex covers and independent sets [10, 11, 13] or matchings [2]. The reader is referred to the surveys [14, 16] for a more complete overview on reconfiguration problems. In this work, we focus on dominating set reconfiguration. Throughout the paper, all the graphs are finite and simple.

Let G = (V, E) be a graph. A *dominating set* of G is a subset of vertices X such that, for every $v \in V$, either $v \in X$ or v has a neighbor in X. A dominating set can be seen as a subset of tokens placed on vertices which dominates the graph. Three types of elementary operations, called *reconfiguration rules*, have been studied for the reconfiguration of dominating sets.

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- The *token addition-removal* rule (TAR) where each operation consists in either removing a token from a vertex, or adding a token on any vertex (keeping a dominating set).
- The *token jumping* rule (TJ) where an operation consists in moving a token from a vertex to any vertex of the graph (keeping a dominating set).
- The *token sliding* rule (TS) where an operation consists in sliding a token from a vertex to an adjacent vertex.

In this paper, we focus on the reconfiguration of dominating sets with the token sliding rule. Note that we authorize (as well as in the other papers on the topic, see [3]) the dominating sets to be multisets. In other words, several tokens can be put on the same vertex. Bonamy et al. observed in [3] that this choice can modify the reconfiguration graph and the set of dominating sets that can be reached from the initial one. More formally, we consider the following problem:

Dominating Set Reconfiguration under Token Sliding (DSR_{TS})

Input: A graph G, two dominating sets D_s and D_t of G.

Output: Does there exist a dominating set reconfiguration sequence from D_s to D_t under the token sliding rule ?

Dominating Set Reconfiguration under Token Sliding. The dominating set reconfiguration problem has been widely studied with the token addition-removal rule. Most of the earlier works focused on the conditions that ensure that the reconfiguration graph is connected in function of several graph parameters, see e.g. [5, 8, 15]. From a complexity point of view, Haddadan et al. [9], proved that the reachability problem is PSPACE-complete under the addition-removal rule, even when restricted to split graphs and bipartite graphs. They also provide linear time algorithms in trees and interval graphs.

More recently, Bonamy et al. [3] studied the dominating set reconfiguration problem under token sliding. They proved that DSR_{TS} is PSPACE-complete, even restricted to split, bipartite or bounded tree-width graphs. On the other hand, they provide polynomial time algorithms for cographs and dually chordal graphs (which contain interval graphs). In their paper, they raise the following question: is it possible to generalize the polynomial time algorithm for interval graphs to circular arc-graphs ?

They also ask if there exists a class of graphs for which the maximum dominating set problem is NPcomplete but its TS-reconfiguration counterpart is polynomial. They propose the class of circle graphs as a candidate.

Our contribution. In this paper, we answer the questions raised in [3]. First, we prove the following:

Theorem 1. DSR_{TS} *is polynomial in circular arc graphs.*

The very high level idea of the proof is as follows. If we fix a vertex of the dominating set then we can unfold the rest of the graph to get an interval graph. We can then use as a black-box the algorithm of Bonamy et al. on interval graphs to determine if we can slide the fixed vertex of the dominating set to some more desirable position.

Our second main result is the following:

Theorem 2. DSR_{TS} *is* PSPACE-*complete in circle graphs.*

This is answering a second question of [3]. The proof is inspired from the proof that DOMINATING SET IN CIRCLE GRAPHS is NP-complete [12] but has to be adapted for the reconfiguration framework.

Both our results and the previously known results about the complexity of DSR_{TS} in graph classes are summarized in Figure 1.

We left open the following question also raised by Bonamy et al. [3]: does there exist a graph class for which MAXIMUM DOMINATING SET is NP-complete but TS-REACHABILITY is polynomial? In the reconfiguration world, such results are not frequent but exist. For instance the existence of a reconfiguration

sequence between two 3-colorings can be decided in polynomial time [6] while finding a 3-coloring is NP-complete.

We also raise the following question: what is the complexity of the DSR_{TS} problem for outerplanar graphs? Outerplanar graphs form a natural subclass of circle graphs, of bounded treewidth graph, and of planar graphs on which the complexity of the problem is PSPACE-complete.

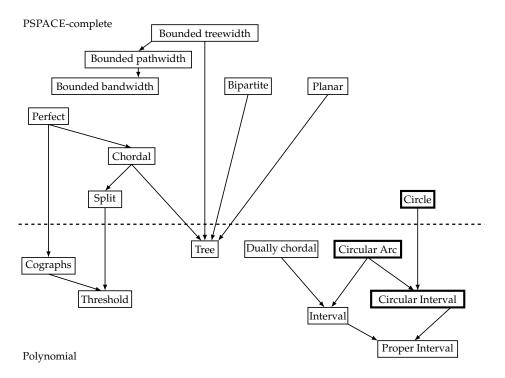


Figure 1: The complexity of DSR_{TS} in several graph classes. The thick rectangles are the results we show in this paper and the other ones are previously known results.

2 Preliminaries

Let G = (V, E) be a graph. Given a vertex $v \in V$, N(v) denotes the open neighborhood of v, i.e. the set $\{y \in V : vy \in E\}$.

A *multiset* is defined as a set with multiplicities. In other words, in a multiset an element can appear several times. The number of times an element appears is the *multiplicity* of this element. The multiplicity of an element that does not appear in the multiset is 0. Let *A* and *B* be multisets. The *union* of *A* and *B*, denoted by $A \cup B$, is the multiset containing only elements of *A* or *B*, and in which the multiplicity of each element is the sum of their multiplicities in *A* and *B*. The *difference* $A \setminus B$ denotes the multiset containing only elements of *A*, and in which the multiplicity of each element is the difference between its multiplicity in *A* and its multiplicity in *B* (if the result is negative then the element is not in $A \setminus B$). By abuse of language, all along this paper, we may refer to multisets as sets.

A *dominating set* D of G is a multiset of elements of V, such that for any $v \in V$, $v \in D$ or there exists $u \in D$ such that $uv \in E$.

Under the token sliding rule, a *move* $v_i \rightsquigarrow v_j$, from a set S_r to a set S_{r+1} , denotes the token sliding operation along the edge $v_i v_j$ from v_i to v_j , i.e. $S_{r+1} = S_r \cup v_j \setminus v_i$. We say that a set *S* is before a set *S'* in a reconfiguration sequence *S* if *S* contains a subsequence starting with *S* and ending with *S'*.

3 A polynomial time algorithm for circular arc-graphs

An *interval graph* G = (V, E) is an intersection graph of intervals of the real line. In other words, the set of vertices is a set of real intervals I and two vertices are adjacent if their corresponding intervals intersect. A *circular arc graph* G = (V, E) is an intersection graph of intervals of a circle. In other words, every vertex is associated an arc A of the circle and there is an edge between two vertices if their two corresponding arcs intersect. By abuse of notation, we refer to the vertices by their image arc. Circular arc graphs strictly contain interval graphs (since long induced cycles are circular arc graphs and not interval graphs). Bonamy et al. proved the following result in [3] that we will use as a black-box:

Theorem 3. [Bonamy et al. [1]] Let G be a connected interval graph, and D_s , D_t be two dominating sets of G of the same size. There always exist a TS-reconfiguration sequence from D_s to D_t .

One can naturally wonder if Theorem 3 can be extended to circular arc graphs. The answer is negative since, for every k, the cycle C_{3k} of length 3k is a circular arc graph and there are only three dominating sets of size exactly k (the ones containing vertices $i \mod 3$ for $i \in \{0, 1, 2\}$) which are pairwise non adjacent for the TS-rule.

However, we prove that we can decide in polynomial time if we can transform one dominating set into another. The remaining of this section is devoted to prove Theorem 1.

Let G = (V, E) be a circular arc graph and D_s, D_t be two dominating sets of G of the same size k.

Assume first that there exists an arc $v \in V$ that contains the whole circle. So A is a dominating set of G and then for any two dominating sets D_s and D_t of G, we can move a token from D_s to v, then move every other other token of D_s to a vertex of D_t (in at most two steps passing through A), and finally move the token on v to the last vertex of D_t . Since a token is on v all along the transformation, all the intermediate steps are indeed dominating sets. So if such an arc exists, there exists a reconfiguration sequence from D_s to D_t .

From now on we assume that no arc contains the whole circle (and that no vertex is dominating the graph). For any arc $v \in V$, the *left extremity* of v, denoted by $\ell(v)$, is the first extremity of v we meet when we follow the circle clockwise, starting from a point outside of v. The other extremity of v is called the *right extremity* and is denoted by r(v).We now construct G_u from G'_u . First remove the vertex u. Note that after this deletion, no arc intersects the open interval $(\ell(u), r(u))$ so the resulting graph is an interval graph. We can unfold it in such a way the first vertex starts at position $\ell(u)$ and the last vertex ends at position r(u) (see Figure 2). We add two new vertices, u' and u'', that correspond to each extremity of u. One has interval $(-\infty, \ell(u)]$ and the other has interval $[r(u), +\infty)$. Since no arc but u' (resp. u'') intersects $(-\infty, \ell(u)]$ (resp. $[r(u), +\infty)$), we can create (n+2) new vertices only adjacent to u' (resp. u''). These 2n + 4 vertices are called the *leaves* of G_u .

Let us first prove the following straightforward lemma.

Lemma 1. Let G be a graph, and u and v be two vertices of G such that $N(u) \subseteq N(v)$. If S is a dominating set reconfiguration sequence in G, and S' is obtained from S by replacing every occurrence of u by v in the dominating sets of S, then S' also is a dominating set reconfiguration sequence in G.

Proof. Every neighbor of u also is a neighbor of v. Thus, replacing u by v in a dominating set keeps the domination of G. Moreover, any move that involves u can be applied if we replace it by v, which gives the result.

In the proof of Theorem 1, we will need the following auxiliary graph G_u (see Figure 2 for an illustration of the construction). Let u be a vertex of G that is maximal by inclusion (no arc strictly contains it). The circular graph G'_u is the graph such that, for every $v \neq u$ which is not contained in u, we create an arc $A_{v'}$ which is the closure of $A_v \setminus A_u^{-1}$. Since u is maximal by inclusion, v' is an arc. We finally add in G'_u the arc of u. Note that the set of edges in G'_u might be smaller than the one of G but any dominating set of G

¹In other words, the arc of v' is the part of the arc of v that is not included in the arc of u. Also note that the fact that A'_v is the closure of that arc ensures that that A_u and $A_{v'}$ intersect.

containing u is a dominating set of G'_u . We now construct G_u from G'_u . First remove the vertex u. Note that after this deletion, no arc intersects the open interval $(\ell(u), r(u))$ so the resulting graph is an interval graph. We can unfold it in such a way the first vertex starts at position $\ell(u)$ and the last vertex ends at position r(u) (see Figure 2). We add two new vertices, u' and u'', that correspond to each extremity of u. One has interval $(-\infty, \ell(u)]$ and the other has interval $[r(u), +\infty)$. Since no arc but u' (resp. u'') intersects $(-\infty, \ell(u)]$ (resp. $[r(u), +\infty)$), we can create (n+2) new vertices only adjacent to u' (resp. u''). These 2n + 4 vertices are called the *leaves* of G_u .

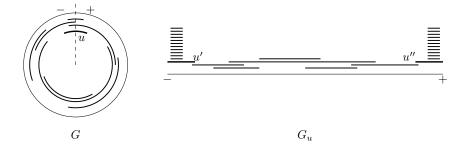


Figure 2: The linear interval graph G_u obtained from the circular arc graph G.

Let us first prove a couple of simple facts about dominating sets of G_u .

Lemma 2. Let D be a dominating set of G such that $u \in D$, and let D_u be the set $D \cup \{u', u''\} \setminus \{u\}$. The set D_u is a dominating set of G_u .

Proof. Every vertex of N(u) in the original graph G is either not in G_u , or is dominated by u' or u''. The neighborhood of all the other vertices have not been modified. Moreover, all the new vertices are dominated since they are all adjacent to u' or u''.

Note that D_u has size |D| + 1.

Lemma 3. The following holds:

- (i) All the dominating sets of G_u of size |D| + 1 contain u' and u''.
- (ii) For every dominating set X of G_u of size |D| + 1, $(X \cap V) \cup \{u\}$ is a dominating set of G of size at most |D|.
- (iii) Every reconfiguration sequence in G_u between two dominating sets D_s , D_t of G_u of size at most |D| + 1 that does not contain any leaf can be adapted into a reconfiguration sequence in G between $(D_s \setminus \{u', u''\}) \cup \{u\}$ and $(D_t \setminus \{u', u''\}) \cup \{u\}$.

Proof. **Proof of (i).** The point (i) holds since there are n + 2 leaves attached to each of u' and u'' and that $|D| \le n$.

Proof of (ii). The vertices u' and u'' only dominate vertices of V dominated by u in G and u' and u'' are in any dominating set of size at most |D| + 1 of G_u by (i). Moreover no edge between two vertices $x, y \in V(G)$ was created in G_u . Thus $(X \cap V) \cup \{u\}$ is a dominating set of G since the only vertices of V(G) that are not in $V(G_u)$ are vertices whose arcs are strictly included in u and then are dominated by u.

Proof of (iii). By Lemma 1, we can assume that there is no token on u' or u'' at any point. We show that we can adapt the transformation. If the move $x \rightsquigarrow y$ satisfies that $x, y \notin \{u', u''\}$ then the same edge exists in G and by (ii), the resulting set is dominating. So we can assume that x or y are u' or u''. We simply have to slide from or to u since N(u') and N(u'') minus the leaves is equal to N(u). Since there is never a token on the leaves, the conclusion follows.

By Lemma 3 and Theorem 3, we immediately obtain the following corollary:

Corollary 1. Let G be a circular interval graph, $u \in V(G)$, and k be an integer. All the k-dominating sets of G containing u are in the same connected component of the reconfiguration graph.

We now have all the ingredients to prove Theorem 1.

Proof of Theorem 1. Let G = (V, E) be a circular arc graph, and let D_s and D_t be two dominating sets of G. Free to slide tokens, we can assume that all the intervals of D_s and D_t are maximal by inclusion. Moreover, by Lemma 1, we can assume that all the vertices of all the dominating sets we will consider are maximal by inclusion. By abuse of notation, we say that in G, an arc v is the *first arc on the left* (resp. *on the right*) of another arc u if the first left extremity of an inclusion-wise maximal arc (of G, or of the stated dominating set) we encounter when browsing the circle counter clockwise (resp. clockwise) from the left extremity of u is the one of v. In interval graphs, we say that an interval v is at *the left* (resp. at *the right*) of an interval u if the left extremity of v is smaller (resp. larger) than the one of u. Note that since the intervals of the dominating sets are maximal by inclusion, the left and right ordering of these vertices are the same. So we can assume that we have a total ordering of the vertices of the dominating sets we are considering.

Let u_1 be a vertex of D_s . Let v be the first vertex at the right of u_1 in D_t . We perform the following algorithm, called the Right Sliding Algorithm. By Lemma 3, all the dominating sets of size $|D_s| + 1$ in G_{u_1} contain u'_1 and u''_1 . Let D'_2 be a dominating set of the interval graph G_{u_1} of size $|D_s| + 1$, such that the first vertex at the right of u'_1 has the smallest left extremity (we can indeed find such a dominating set in polynomial time). By Theorem 3, there exists a transformation from $(D_s \cup \{u'_1, u''_1\}) \setminus \{u_1\}$ to D'_2 in G_{u_1} . And thus by Lemma 3, there exists a transformation from D_s to $D_2 := (D'_2 \cup \{u_1\}) \setminus \{u'_1, u''_1\}$ in G. We apply this transformation. Informally speaking, this transformation has permitted to move the token at the left of u closest from u that will hopefully permit to push the token on u to the right.

Now, we fix all the vertices of D_2 but u_1 , and we try to slide the token on u_1 to its right. If we can push it on a vertex at the right of v, we can in particular push it on v (since v is maximal by inclusion) and keep a dominating set. So we set $u_2 = v$ if we can reach v or the rightmost possible vertex maximal by inclusion we can reach otherwise. We now repeat these operations with u_2 instead of u_1 , i.e. we apply a reconfiguration sequence towards a dominating set of G in which the first vertex on the left of u_2 is the closest to u_2 , then try to slide u_2 to the right, onto u_3 . We repeat these operations until $u_i = u_{i+1}$ (i.e. we cannot move to the right anymore) or until $u_i = v$. Let u_1, \ldots, u_ℓ be the resulting sequence of vertices. Note that this algorithm is indeed polynomial since after at most n steps we have reached v or reached a fixed point.

We can similarly define the Left Sliding Algorithm by replacing the leftmost dominating set of G_{u_i} by the rightmost, and then slide u_i to the left for any *i*. We stop when we cannot slide to the left anymore, or when $u_i = v'$, where v' is the first vertex at the left of u_1 in D_t . Let u'_{ℓ} be the last vertex of the sequence of vertices given by the Left Sliding Algorithm.

To conclude the proof we simply have to show the following claim:

Claim 1. There exists a transformation from D_s to D_t if and only if $u_\ell = v$ or $u'_\ell = v'$.

Proof. Firstly, if $u_{\ell} = v$, then Corollary 1 ensures that there exists a transformation from D_{ℓ} to D_t and thus from D_s to D_t , and similarly if $u'_{\ell} = v'$.

Let us now prove the converse direction. If $u_{\ell} \neq v$ and $u'_{\ell} \neq v'$, assume for contradiction that there exists a transformation sequence *S* from D_s to D_t . By Lemma 1 we can assume that all the vertices in any dominating set of *S* are maximal by inclusion.

Let us consider the first dominating set *C* of *S* where the token initially on u_1 is at the right of u_ℓ in *G*, or at the left of u'_ℓ in *G*. Such a dominating set exists no token of D_t is between $u'_{\ell'}$ and u_ℓ . Let us denote by *C'* the dominating before *C* in the sequence and $x \rightsquigarrow y$ the move from *C* to *C'*. By symmetry, we can assume that *y* is at the right of u_ℓ . Note that *x* is at the left of u_ℓ . Note that $C'' = C \setminus \{x\} \cup \{u_\ell\}$ is a dominating set of *G* since *C* and $C' = C \setminus \{x\} \cup \{y\}$ are dominating sets and u_ℓ is between *x* and *y*.

So $C \setminus \{x\} \cup \{u_{\ell}, u_{\ell}'\}$ is a dominating set of $G_{u_{\ell}}$ and then for C'' it was possible to move the token on u_{ℓ} to the right, a contradiction with the fact that u_{ℓ} was a fixed point.

4 **PSPACE-hardness for Circle Graphs**

A *circle graph* G = (V, E) is an intersection graph of chords of a circle (i.e. segments between two points of a circle). Let *C* be a circle. Equivalently, we can associate to each vertex of a circle graph two points of *C*. And there is an edge between two vertices if the chords between their pair of points intersect. Again equivalently, a circle graph can be represented on the real line. We associate to each vertex an interval of the real line; and there is an edge between two vertices if their intervals intersect but do not overlap. In this section, we will use the last representation of circle graphs. For every interval *I*, $\ell(I)$ will denote the *left extremity* of *I*, and r(I) the *right extremity* of *I*.

The goal of this section is to show that DSR_{TS} is PSPACE-complete in circle graphs. We provide a polynomial time reduction from SATR to DSR_{TS} . This reduction is inspired from one used in [12] to show that the minimum dominating set problem is NP-complete on circle graphs but has to be adapted in the reconfiguration framework. The SATR problem is defined as follows:

SATISFIABILITY RECONFIGURATION (SATR)

Input: A Boolean formula *F* in conjunctive normal form (conjunction of clauses), two variable assignments A_s and A_t that satisfy *F*.

Output: Does there exist a reconfiguration sequence from A_s to A_t that keeps F satisfied, where the operation consists in a *variable flip*, i.e. the change of the assignment of exactly one variable from x = 0 to x = 1, or conversely ?

Let (F, A_s, A_t) be an instance of the SATR problem. Let $x_1 \ldots, x_n$ be the variables of the boolean formula F. Since F is in conjunctive normal form, it is a conjunction of *clauses* c_1, \ldots, c_m which are disjunctions of literals. A *literal* is a variable or the negation of a variable, and we denote by $x_i \in c_j$ (resp. $\overline{x_i} \in c_j$) the fact that x_i (resp. the negation of x_i) is a literal of c_j . Since duplicating clauses does not modify the satisfiability of a formula, we can assume without loss of generality that m is a multiple of 4. We can also assume that for every $i \leq n$ and $j \leq m$, and that, for every i, j, x_i or $\overline{x_i}$ are not in c_j (since otherwise the clause is satisfied for any possible assignment and can be removed from the boolean formula).

4.1 The reduction.

Let us construct an instance $(G_F, D_F(A_s), D_F(A_t))$ of the DSR_{TS} problem from (F, A_s, A_t) . We start by constructing the circle graph G_F from F. All along this construction, we repeatedly refer to real number as *points*. We say that a point p is *at the left* of a point q (or q is *at the right* of p) if p < q. We say that p is *just at the left* of q, (or q is *just at the right* of p) if p is at the right of p) if p is at the left of q, and no interval defined so far has an extremity in [p, q]. Finally, we say that an interval I frames a set of points P if $\ell(I)$ is just at the left of the minimum of P and r(I) is just at the right of the maximum of P.

One can easily check that by adding an interval that frames one extremity of the interval of a vertex u of a graph H, we add one vertex to H which is only connected to u. So:

Remark 1. If H is a circle graph and u is a vertex of H, then the graph H plus a new vertex only connected to u is circle graph.

We construct G_F step by step. The construction of G_F is quite technical and will be performed step by step. The construction is inspired from [12]. In [12], the authors have decided to give the coordinates of the endpoints of all the intervals. For the sake of readability, we think that it is easier to only give the relative positions of the intervals between them.

Each step consists in creating new intervals, and in giving their relative positions regarding to the previously constructed intervals. We also outline some of the edges and non edges in G_F that have an impact on the upcoming proofs². Figures 3, 4 and 5 will illustrate the positions of the intervals of G_F .

For each variable x_i , we create *m* base intervals B_j^i where $1 \le j \le m$. The base intervals B_j^i are pairwise disjoint for any *i* and *j*, and are ordered by increasing *i*, then increasing *j* for a same *i*.

²Some adjacencies between intervals that will be anyway dominated for some reasons that will become clear later on will not be discussed.

For each variable x_i , we then create $\frac{m}{2}$ intervals X_j^i called the *positive bridge intervals* of x_i , and $\frac{m}{2}$ intervals \overline{X}_j^i called the *negative bridge intervals* of x_i , where $1 \le j \le \frac{m}{2}$. A *bridge interval* is a positive or a negative bridge interval. Let us give the positions of these intervals. They are illustrated in Figure 3.

Figure 3: The base, positive and negative bridge intervals obtained with n = 2 and m = 8.

Let q be such that m = 4q. For every i and every $0 \le r < q$, the interval \overline{X}_{2r+1}^i starts just at the right of $\ell(B_{4r+2}^i)$ and ends just at the right of $\ell(B_{4r+2}^i)$, and \overline{X}_{2r+2}^i starts just at the right of $\ell(B_{4r+2}^i)$ and ends just at the right of $\ell(B_{4r+4}^i)$. The interval X_1^i starts just at the left of $r(B_1^i)$ and ends just at the left of $r(B_2^i)$. For every $1 \le r < q$, the interval X_{2r}^i starts just at the left of $r(B_{4r-1}^i)$ and ends just at the left of $r(B_{4r+1}^i)$, and X_{2r+1}^i starts just at the left of $r(B_{4r+1}^i)$, and ends just at the left of $r(B_{4r+1}^i)$, and ends just at the left of $r(B_{4r+1}^i)$, and X_{2r+1}^i starts just at the left of $r(B_{4r}^i)$ and ends just at the left of $r(B_{4r+2}^i)$. Finally, $X_{\frac{m}{2}}^i$ starts just at the left of $r(B_{m-1}^i)$ and ends just at the left of $r(B_m^i)$.

Let us outline some of the edges induced by these intervals. Base intervals are pairwise non adjacent. Moreover, every positive (resp. negative) bridge interval is incident to exactly two base intervals; And all the positive (resp. negative) bridge intervals of x_i are incident to pairwise distinct base intervals. In particular, the positive (resp. negative) bridge intervals dominate the base intervals; And every base interval is adjacent to exactly one positive and one negative bridge interval. All the positive (resp. negative) bridge intervals but X_1^i and $X_{\frac{in}{2}}^{\frac{in}{2}}$ have exactly one other positive (resp. negative) bridge interval neighbor. Finally, for every *i*, every negative bridge interval \overline{X}_j^i has exactly two positive bridge interval neighbors which are X_{j-1}^i and X_j^i except for \overline{X}_1^i which does not have any for any *i*. Note that a bridge interval of x_i is not adjacent to a bridge interval or a base interval of x_j for $j \neq i$.

Now for any clause c_j , we create two identical *clause intervals* C_j and C'_j . In this paper, we consider that two identical intervals do overlap, so that C_j and C'_j are not adjacent. The clause intervals C_j are pairwise disjoint and ordered by increasing j, and we have $\ell(C_1) > r(B_m^n)$. Thus, they are not adjacent to any interval constructed so far.

For every j such that x_i is in the clause c_j , we create four intervals T_j^i , U_j^i , V_j^i and W_j^i , called the *positive path intervals* of x_i ; and for every j such that $\overline{x_i}$ is in the clause c_j , we create four intervals \overline{T}_j^i , \overline{U}_j^i , \overline{V}_j^i and \overline{W}_j^i , called the *negative path intervals* of x_i . These intervals are represented in Figure 4. In order to give a better representation of the relative position of the extremities, a zoom on that part of the graph is proposed in Figure 5. The interval T_j^i frames the right extremity of B_j^i and the extremity of the positive bridge interval that belongs to B_j^i . The interval \overline{T}_j^i frames the left extremity of B_j^i and the extremity of the negative bridge interval that belongs to B_j^i . The interval U_j^i starts just at the left of $r(T_j^i)$, the interval \overline{U}_j^i starts just at the right of $l(\overline{T}_j^i)$, and they both end between the right of the last base interval of the variable x_i and the left of the next base or clause interval. We moreover construct the intervals U_j^i (resp. \overline{U}_j^i) in such a way $r(U_j^i)$ (resp. $r(\overline{U}_j^i)$) is increasing when j is increasing. In other words, the U_j^i (resp. \overline{U}_j^i) are pairwise adjacent. The interval V_j^i (resp. \overline{V}_j^i) frames the right extremity of U_j^i (resp. \overline{U}_j^i) and the interval W_j^i (resp. \overline{W}_j^i) starts just at the left of $r(V_j^i)$ (resp. $r(\overline{V}_j^i)$) and ends in an arbitrary point of C_j . Moreover, for any $i \neq i'$, W_j^i (resp.

 $\overline{W_j^i}$) and $W_j^{i'}$ (resp. $\overline{W_j^{i'}}$) end on the same point of C_j . This ensures that they overlap and are therefore not adjacent.

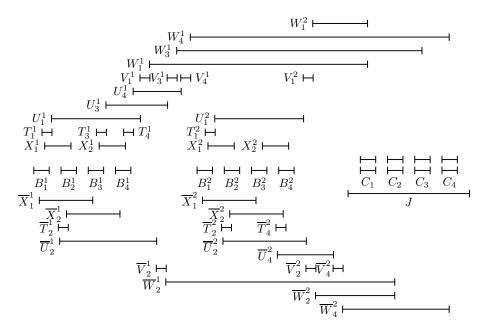


Figure 4: The intervals obtained for the formula $F = (x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2}) \wedge (x_1) \wedge (\overline{x_2} \vee x_1)$ with m = 4 clauses and n = 2 variables. The dead-end intervals and the pending intervals are not represented here.

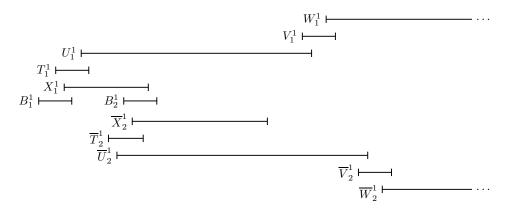


Figure 5: A zoom on some intervals of the variable x_1 .

A *path interval* is a positive or a negative path interval. The *intervals of* x_i are the base, bridge and path intervals of x_i . The T intervals of x_i refers to the intervals T_j^i for any j. The \overline{T} , U, \overline{U} , V, \overline{V} , W and \overline{W} intervals of x_i are defined similarly.

Let us outline some neighbors of the path intervals. The neighborhood of every clause interval C_j is the set of intervals W_j^i with $x_i \in c_j$ and intervals \overline{W}_j^i with $\overline{x_i} \in c_j$. Since V_j^i spans the left extremity of W_j^i and the right extremity of U_j^i and since no interval starts or ends between these two points, the interval V_j^i is only adjacent to U_j^i and \overline{W}_j^i . Similarly \overline{V}_j^i is only adjacent to \overline{U}_j^i and \overline{W}_j^i . Moreover, T_j^i is only adjacent to B_j^i , \overline{U}_j^i and one positive bridge interval (the same one that is adjacent to B_j^i), and \overline{T}_j^i is only adjacent to B_j^i , \overline{U}_j^i and one negative bridge interval (the same one that is adjacent to B_j^i). Moreover, since U_j^i and W_j^i are not

adjacent, B_j^i , T_j^i , U_j^i , V_j^i , W_j^i and C_j induce a path, and since \overline{U}_j^i and \overline{W}_j^i are not adjacent, B_j^i , \overline{T}_j^i , \overline{U}_j^i , \overline{V}_j^i , \overline{W}_j^i , \overline{W}_j

Now, for every bridge interval and every U, \overline{U} , W and \overline{W} interval, we create a *dead-end interval*, that is only adjacent to it. Remark 1 ensures that it can be done while keeping a circle graph. Then, for any deadend interval, we create 6mn pending intervals that are each only adjacent to it. Again, Remark 1 ensures that the resulting graph is a circle graph. Informally speaking, since the dead-end intervals have a lot of pending intervals, they will be forced to be in any dominating set of size at most 6mn. Thus, in any dominating set, we will know that bridge, U, \overline{U} , W and \overline{W} intervals (as well as dead-end an pending vertices) are already dominated. So the other vertices in the dominating set will only be there to dominate the other vertices of the graph, which are called the *important vertices*.

Finally, we create a *junction interval* J, that frames $\ell(C_1)$ and $r(C_m)$. By construction, it is adjacent to every W or \overline{W} interval, and to no other interval. This completes the construction of the graph G_F .

4.2 Basic properties of G_F

Let us first give a couple of properties satisfied by G_F . The following lemma will be used to guarantee that any token can be moved to any vertex of the graph as long as the rest of the tokens form a dominating set.

Lemma 4. The graph G_F is connected.

Proof. Let x_i be a variable. Let us first prove that the intervals of x_i are in the same connected component of G_F . (Recall that they are the base, bridge and path intervals of x_i). Firstly, for any j such that $x_i \in C_j$ (resp. $\overline{x_i} \in C_j$), $B_j^i T_j^i U_j^i V_j^i W_j^i$ (resp. $B_j^i \overline{T}_j^i \overline{U}_j^i \overline{V}_j^i \overline{W}_j^i$) is a path of G_F . Since every base interval of x_i is adjacent to a positive and a negative bridge interval of x_i , it is enough to show that all the bridge intervals of x_i are in the same connected component. Since for every $j \ge 2$, \overline{X}_j^i is adjacent to X_{j-1}^i and X_j^i , we know that $X_1^i \overline{X}_2^i, X_2^i \dots \overline{X}_{\frac{m}{2}}^i X_{\frac{m}{2}}^i$ is a path of G_F . Moreover, \overline{X}_1^i is adjacent to \overline{X}_2^i . So all the intervals of x_i are in the same connected component of G_F .

Now, since the junction interval J is adjacent to every W and \overline{W} interval (and that each variable appears in at least one clause), J is in the connected component of all the path variables, so the intervals of x_i and $x_{i'}$ are in the same connected component for every $i \neq i'$. Since each clause contains at least one variable, C_j is adjacent to at least one interval W_j^i or \overline{W}_j^i . Finally, each dead-end interval is adjacent to a bridge interval or a U, \overline{U}, W or \overline{W} interval, and each pendant interval is adjacent to a dead-end interval. Therefore, G_F is connected.

For any variable assignment A of F, let $D_F(A)$ be the set of intervals of G_F defined as follows. The junction interval J belongs to $D_F(A)$ and all the dead-end intervals belong to $D_F(A)$. For any variable x_i such that $x_i = 1$ in A, the positive bridge, W and \overline{U} intervals of x_i belong to $D_F(A)$. Finally, for any variable x_i such that $x_i = 0$ in A, the negative bridge, \overline{W} and U intervals of x_i belong to $D_F(A)$. The multiplicity of each of these intervals in $D_F(A)$ is one. Thus, we have $|D_F(A)| = \frac{3mn}{2} + 3\sum_{i=1}^{n} \ell_i + 1$ where for any variable x_i , ℓ_i is the number of clauses that contain x_i or $\overline{x_i}$.

Lemma 5. If A satisfies F, then $D_F(A) \setminus J$ is a dominating set of G_F .

Proof. Since every dead-end interval belongs to $D_F(A) \setminus J$, every pending and dead-end interval is dominated, as well as every bridge, U, \overline{U}, W and \overline{W} interval. Since for each variable x_i , the positive (resp. negative) bridge intervals of x_i dominate the base intervals of x_i , the base intervals are dominated. Moreover, the positive (resp. negative) bridge intervals of x_i and the U (resp. \overline{U}) intervals of x_i both dominate the T (resp. \overline{T}) intervals of x_i . Thus, the T and \overline{T} intervals are all dominated. Moreover, for any variable x_i , the U and W (resp. \overline{U} and \overline{W}) intervals of x_i both dominate the V (resp. \overline{V}) intervals of x_i . Thus, the T and \overline{T} intervals are all dominated. Moreover, for any variable x_i , the U and W (resp. \overline{U} and \overline{W}) intervals of x_i both dominate the V (resp. \overline{V}) intervals of x_i . Thus, the V and \overline{V} intervals are all dominated. Finally, since A satisfies F, each clause has at least one of its literal in A. Thus, each C_j and C'_j has at least one adjacent interval W_j^i or \overline{W}_j^i in $D_F(A) \setminus J$ and are therefore dominated by it, as well as the junction interval.

Before continuing further, let us prove a few results that are of importance in our proof. Let $K := \frac{3mn}{2} + 3\sum_{i=1}^{n} \ell_i + 1$. Since the number 6mn of leaves attached on each dead-end interval is strictly more than K (as $\ell_i \leq m$), the following holds.

Remark 2. Any dominating set of size at most K contains all the $(mn + 2\sum_{i=1}^{n} \ell_i)$ dead-end intervals.

Any dominating set of size *K* contains all the dead-end vertices. And then all the pending, dead-end, bridge, U, \overline{U}, W and \overline{W} intervals are dominated. So we will simply have to focus on the domination of base, $T, \overline{T}, V, \overline{V}$ and junction intervals (i.e. the so-called important intervals).

Lemma 6. If D is a dominating set of G, then for any variable x_i , D contains at least ℓ_i intervals that dominate the V and \overline{V} intervals of x_i , and at least $\frac{m}{2}$ intervals that dominate the base intervals of x_i . Moreover, these two sets of intervals are disjoint, and they are intervals of x_i .

Proof. For any variable x_i , each interval V_j^i (resp. \overline{V}_j^i) can only be dominated by U_j^i , V_j^i or W_j^i (resp. \overline{U}_j^i , \overline{V}_j^i or \overline{W}_j^i). Indeed V_j^i spans the left extremity of W_j^i and the right extremity of U_j^i and since no interval starts or ends between these two points, the interval V_j^i is only adjacent to U_j^i and W_j^i . And similarly \overline{V}_j^i is only adjacent to \overline{U}_j^i and \overline{W}_j^i . Thus, at least ℓ_i intervals dominate the V and \overline{V} intervals of x_i , and they are intervals of x_i . Moreover, only the base, bridge, T and \overline{T} intervals of x_i are adjacent to the base intervals. Since each bridge interval is adjacent to two base intervals, and each T and \overline{T} interval of x_i is adjacent to one base interval of x_i , D must contain at least $\frac{m}{2}$ of such intervals to dominate the m base intervals.

Remark 2 and Lemma 6 imply that any dominating set *D* of size *K* contains $(mn + 2\sum_{i=1}^{n} \ell_i)$ dead-end intervals, as well as $(\ell_i + \frac{m}{2})$ intervals of x_i for any variable x_i . Since $K = \frac{3mn}{2} + 3\sum_{i=1}^{n} \ell_i + 1$, this leaves only one remaining token in *D*. Thus, for any variable x_i but at most one, there are exactly $(\ell_i + \frac{m}{2})$ intervals of x_i in *D*. If there exists a variable x_k such that there are more $(\ell_i + \frac{m}{2})$ intervals of x_k in *D*, then there are exactly $(\ell_k + \frac{m}{2} + 1)$ intervals of x_k in *D*, and we call this variable the *moving variable* of *D*, denoted by mv(D).

For any variable x_i , we denote by X_i the set of positive bridge variables of x_i and by $\overline{X_i}$ the set of negative bridge variables of x_i . Similarly, we denote by W_i the set of W variables of x_i and by $\overline{W_i}$ the set of \overline{W} variables of x_i . Let us now give some precision about the intervals of x_i that belong to D.

Lemma 7. If D is a dominating set of size K, then for any variable $x_i \neq mv(D)$, either $X_i \subseteq D$ and $\overline{X_i} \cap D = \emptyset$, or $\overline{X_i} \subseteq D$ and $X_i \cap D = \emptyset$.

Proof. Since $x_i \neq mv(D)$, there are exactly $\ell_i + \frac{m}{2}$ variables of x_i in D. Thus, by Lemma 6, exactly $\frac{m}{2}$ intervals of x_i in D dominate the bridge intervals of x_i . Only the bridge, T and \overline{T} intervals of x_i are adjacent to the base intervals. Moreover, bridge intervals are adjacent to two base intervals and T or \overline{T} intervals are adjacent to only one. Since there are m base intervals of x_i , each interval of D must dominate a pair of base intervals (or none of them). So these intervals of D should be some bridge intervals of x_i .

Note that, by cardinality, each pair of bridge intervals of D must dominate pairwise disjoint base intervals. Let us now show by induction that these bridge intervals are either all the positive bridge intervals, or all the negative bridge intervals. We study two cases: either $X_1^i \in D$, or $X_1^i \notin D$.

Assume that $X_1^i \in D$. In D, X_1^i dominates B_1^i and B_2^i . Thus, since \overline{X}_1^i dominates B_1^i and \overline{X}_2^i dominates B_2^i , none of $\overline{X}_1^i, \overline{X}_2^i$ are in D (since their neighborhood in the set of base intervals is not disjoint with X_1^i). But B_3^i (resp B_4^i) is only adjacent to \overline{X}_1^i and X_2^i (resp. \overline{X}_2^i and X_3^i). Thus both X_2^i, X_3^i are in D. Suppose now that for a given j such that j is even and $j \leq \frac{m}{2} - 2$, we have $X_j^i, X_{j+1}^i \in D$. Then, since a base interval dominated by X_j^i (resp. X_{j+1}^i) also is dominated by \overline{X}_{j+1}^i (resp. \overline{X}_{j+2}^i), the intervals $\overline{X}_{j+1}^i, \overline{X}_{j+2}^i$ are not in D. But there is a base interval adjacent only to \overline{X}_{j+1}^i and X_{j+2}^i (resp. \overline{X}_{j+2}^i and X_{j+3}^i if $j \neq \frac{m}{2} - 2$, or \overline{X}_{j+2}^i and X_{j+2}^i if $j = \frac{m}{2} - 2$). Therefore, if $j + 2 < \frac{m}{2}$ we have $X_{j+2}^i, X_{j+3}^i \in D$, and $X_{\frac{m}{2}}^i \in D$. By induction, if $X_1^i \in D$ then each of the $\frac{m}{2}$ positive bridge intervals belong to D and thus none of the negative bridge intervals do. Assume now that $X_1^i \notin D$. Then, to dominate B_1^i and B_2^i , we must have $\overline{X}_1^i, \overline{X}_2^i \in D$. Let us show that if for a given odd j such that $j \leq \frac{m}{2} - 3$ we have $\overline{X}_j^i, \overline{X}_{j+1}^i \in D$, then $\overline{X}_{j+2}^i, \overline{X}_{j+3}^i \in D$. Since \overline{X}_j^i (resp. \overline{X}_{j+1}^i) dominates base intervals also dominated by X_{j+1}^i (resp. X_{j+2}^i), we have $X_{j+1}^i, X_{j+2}^i \notin D$. But there exists a base interval only adjacent to X_{j+1}^i and \overline{X}_{j+2}^i (resp. X_{j+2}^i and \overline{X}_{j+3}^i). Thus, $\overline{X}_{j+2}^i, \overline{X}_{j+3}^i \in D$. By induction, if $X_1^i \notin D$ then each of the $\frac{m}{2}$ negative bridge intervals belong to D. Thus, none of the positive bridge intervals belong to D.

Lemma 8. If D is a dominating set of size K, then for any variable $x_i \neq mv(D)$, if $X_i \subseteq D$ then $\overline{W_i} \cap D = \emptyset$, otherwise $W_i \cap D = \emptyset$.

Proof. By Lemma 7, *D* either contains X_i or contains $\overline{X_i}$.

If $X_i \subset D$, Lemma 7 ensures that $\overline{X}_i \cap D = \emptyset$. So the intervals \overline{T}_j^i have to be dominated by other intervals.

By Lemma 6, ℓ_i intervals must dominate the V and \overline{V} intervals of x_i . Since no interval dominates two of them, each \overline{T}_j^i has to be dominated by an interval that is also dominating a V or \overline{V} interval. The only interval that dominates both \overline{T}_j^i and a V or \overline{V} interval is \overline{U}_j^i . So all the \overline{U} intervals are in D and $\overline{W} \cap D = \emptyset$ (since the only V or \overline{V} interval dominated by a \overline{W} interval is a \overline{V} interval, which is already dominated).

Similarly if $\overline{X_i} \subset D$, Lemma 7 ensures that $X_i \cap D = \emptyset$. So the intervals T_j^i have to be dominated by other intervals. And one can prove similarly that these intervals should be the U intervals and then the W intervals are not in D.

4.3 Safeness of the reduction.

Let (F, A_s, A_t) be an instance of SATR, and let $D_s = D_F(A_s)$ and $D_t = D_F(A_t)$. By Lemma 5, (G_F, D_s, D_t) is an instance of DSR_{TS}. We can now show the first direction of our reduction.

Lemma 9. If (F, A_s, A_t) is a yes-instance of SATR, then (G_F, D_s, D_t) is a yes-instance of DSR_{TS}.

Proof. Let (F, A_s, A_t) be a yes-instance of SATR, and let $S = \langle A_1 := A_s, \ldots, A_\ell := A_t \rangle$ be the reconfiguration sequence from A_s to A_t . We construct a reconfiguration sequence S' from D_s to D_t by replacing any flip of variable $x_i \rightsquigarrow \overline{x_i}$ of S from A_r to A_{r+1} by the following sequence of token slides from $D_F(A_r)$ to $D_F(A_{r+1})^3$.

- We perform a sequence of slides that moves the token on *J* to *X*ⁱ₁. By Lemma 4, *G_F* is connected, and by Lemma 5, *D_F(A_r) \ J* is a dominating set. So any sequence of moves along a path from *J* to *X*ⁱ₁ keeps a dominating set.
- For any j such that $x_i \in C_j$, we first move the token from W_j^i to V_j^i then from V_j^i to U_j^i . Let us show that this keeps G_F dominated. The important intervals that can be dominated by W_j^i are V_j^i , C_j , and J. The vertex V_j^i is dominated anyway during the sequence since it is also dominated by V_j^i and U_j^i . Moreover, since $x_i \rightsquigarrow \overline{x_i}$ keeps F satisfied, each clause containing x_i has a literal different from x_i that also satisfies the clause. Thus, for each C_j such that $x_i \in C_j$, there exists an interval $W_j^{i'}$ or $\overline{W}_j^{i'}$, with $i' \neq i$, that belongs to $D_F(A_r)$, and then dominates both C_j and J during these two moves.
- For *j* from 1 to $\frac{m}{2} 1$, we apply the move $X_j^i \rightsquigarrow \overline{X}_{j+1}^i$. This move is possible since X_j^i and \overline{X}_{j+1}^i are neighbors in G_F . Let us show that this move keeps a dominating set. For j = 1, the important intervals that are dominated by X_1^i are B_1^i , B_2^i , and T_1^i . Since U_1^i is in the current dominating set (by the second point), T_1^i is dominated. Moreover B_1^i is dominated by \overline{X}_1^i , and B_2^i is a neighbor of \overline{X}_2^i .

 $^{{}^{3}}A \overline{x_{i}} \rightsquigarrow x_{i}$ consists in applying the converse of this sequence.

Thus, $X_1^i \rightsquigarrow \overline{X}_2^i$ maintains a dominating set. For $2 \le j \le \frac{m}{2} - 1$, the important intervals that are dominated by X_j^i are B_k^i , B_{k-2}^i and T_j^i where k = 2j + 1 if j is even and k = 2j otherwise. Again T_j^i is dominated by the U intervals. Moreover B_{k-2}^i is dominated by \overline{X}_{j-1}^i (on which there is a token since we perform this sequence for increasing j), and B_k^i is also dominated by \overline{X}_{j+1}^i .

- For any *j* such that *x_i* ∈ *C_j*, we move the token from *U*ⁱ_j to *V*ⁱ_j and then from *V*ⁱ_j to *W*ⁱ_j. The important intervals dominated by *U*ⁱ_j are the intervals *T*ⁱ_j, *V*ⁱ_j. But *T*ⁱ_j is dominated by a negative bridge interval, and *V*ⁱ_j stays dominated by *V*ⁱ_j then *W*ⁱ_j.
- The previous moves lead to the dominating set (D_F(A_{r+1}) \ J) ∪ Xⁱ/_m. We finally perform a sequence of moves that slide the token on Xⁱ/_m to J. It can be done since Lemma 4 ensures that G_F is connected. And all along the transformation, we keep a dominating set by Lemma 5. As wanted, it leads to the dominating set D_F(A_{r+1}).

We now prove the other direction of the reduction. Let us prove the following lemma.

Lemma 10. If there exists a reconfiguration sequence S from D_s to D_t , then there exists a reconfiguration sequence S' from D_s to D_t such that for any two adjacent dominating sets D_r and D_{r+1} of S', if both D_r and D_{r+1} have a moving variable, then it is the same one.

Proof. Assume that, in *S*, there exist two adjacent dominating sets D_r and D_{r+1} such that both D_r and D_{r+1} have a moving variable, and $mv(D_r) \neq mv(D_{r+1})$. Let us modify slightly the sequence in order to avoid this move.

Since D_r and D_{r+1} are adjacent in S, we have $D_{r+1} = D_r \cup v \setminus u$, where uv is an edge of G_F . Since $mv(D_r) \neq mv(D_{r+1})$, u is an interval of $mv(D_r)$, and v an interval of $mv(D_{r+1})$. By construction, the only edges of G_F between intervals of different variables are between their $\{W, \overline{W}\}$ intervals. Thus, both u and v are W or \overline{W} intervals and, in particular they are adjacent to the junction interval J. Moreover, the only important intervals that are adjacent to u (resp. v) are the V or \overline{V} intervals of the same variable as u, W or \overline{W} intervals, clause intervals, or the junction interval J. Since u and v are adjacent, and since they are both W or \overline{W} intervals, they cannot be adjacent to the same clause interval. But the only intervals that are potentially not dominated by $D_r \setminus u = D_{r+1} \setminus v$ should be dominated both by u in D_r and by v in D_{r+1} . So these intervals are included in the set of W or \overline{W} intervals and the junction interval, which are all dominated by J. Thus, $D_r \cup J \setminus u$ is a dominating set of G_F . Therefore, we can add in S the dominating set $D_r \cup J \setminus u$ between D_r and D_{r+1} . This intermediate dominating set has no moving variable. By repeating this procedure while there are adjacent dominating sets in S with different moving variables, we obtain the desired reconfiguration sequence S'.

Lemma 11. If (G_F, D_s, D_t) is a yes-instance of DSR_{TS}, then (F, A_s, A_t) is a yes-instance of SATR.

Proof. Let (G_F, D_s, D_t) be a yes-instance of DSR_{TS}. There exists a reconfiguration sequence S' from D_s to D_t . Moreover, by Lemma 10, we can assume that for any two adjacent dominating sets D_r and D_{r+1} of S', if both D_r and D_{r+1} have a moving variable, then it is the same one.

Let us construct a reconfiguration sequence S from A_s to A_t . To any dominating set D of G_F , we associate a variable assignment A(D) of F defined as follows. For any variable $x_i \neq mv(D)$, either $X_i \subset D$ or $\overline{X}_i \subset D$ by Lemma 7. If $X_i \subset D$ then we set $x_i = 1$. Otherwise, we set $x_i = 0$. Let x_k be such that $mv(D) = x_k$ if it exists. If there exists a clause interval C_j such that $W_j^k \in D$, and if for any $x_i \neq x_k$ with $\overline{x_i} \in c_j$, we have $\overline{X_i} \subset D$, then we set $x_k = 1$. Otherwise $x_k = 0$.

Let *S* be the sequence of assignments obtained by replacing in *S'* any dominating set *D* by the assignment A(D). In order to conclude, we must show that the assignments associated to D_s and D_t are precisely

 A_s and A_t . Moreover, for every dominating set D, the assignment associated to D has to satisfy F. Finally, for every move in G_F , we must be able to associate a (possibly empty) variable flip. Let us first show a useful claim, then proceed with the end of the proof.

Claim 2. For any consecutive dominating sets D_r and D_{r+1} and any variable x_i that is not the moving variable of D_r nor D_{r+1} , the value of x_i is identical in $A(D_r)$ and $A(D_{r+1})$.

Proof. Lemma 7 ensures that for any x_i such that $x_i \neq mv(D_r)$ and $x_i \neq mv(D_{r+1})$, either $X_i \subset D_r$ and $\overline{X_i} \cap D_r = \emptyset$ or $\overline{X_i} \subset D_r$ and $X_i \cap D_r = \emptyset$, and the same holds in D_{r+1} . Since the number of positive and negative bridge intervals is at least 2 (since by assumption m is a multiple of 4), and D_{r+1} is reachable from D_r in a single step, either both D_r and D_{r+1} contain X_i , or both contain $\overline{X_i}$. Thus, by definition of A(D), for any variable x_i such that $x_i \neq mv(D_r)$ and $x_i \neq mv(D_{r+1})$, x_i has the same value in $A(D_r)$ and $A(D_{r+1})$.

Claim 3. We have $A(D_s) = A_s$ and $A(D_t) = A_t$.

Proof. By definition, $D_s = D_F(A_s)$ and thus D_s contains the junction interval, which means that it does not have any moving variable. Moreover, D_s contains X_i for any variable x_i such that $x_i = 1$ in A_s and $\overline{X_i}$ for any variable x_i such that $x_i = 0$ in A_s . Therefore, for any variable x_i , $x_i = 1$ in A_s if and only if $x_i = 1$ in $A(D_s)$. Similarly, $A(D_t) = A_t$.

Claim 4. For any dominating set D of S', A(D) satisfies F.

Proof. Since the clause intervals are only adjacent to W and \overline{W} intervals, they are dominated by them, or by themselves in D. But only one clause interval can belong to D. Thus, for any clause interval C_j , if $C_j \in D$, then C'_j must be dominated by a W or a \overline{W} interval, that also dominates C_j . So in any case, C_j is dominated by a W or a \overline{W} interval. We study four possible cases and show that in each case, c_j is satisfied by A(D).

If C_j is dominated in D by an interval W_j^i , where $x_i \neq mv(D)$, then by Lemmas 7 and 8, $X_i \subset D$ and by definition of A(D), $x_i = 1$. Since W_j^i exists, it means that $x_i \in c_j$, thus c_j is satisfied by A(D).

Similarly, if C_j is dominated in D by an interval \overline{W}_j^i , where $x_i \neq mv(D)$, then by Lemmas 7 and 8, $\overline{X_i} \subset D$. So $x_i = 0$. Since \overline{W}_j^i exists, $\overline{x_i} \in c_j$, and therefore c_j is satisfied by A(D).

If C_j is only dominated by W_j^k in D, where $x_k = mv(D)$. Then, if there exists $x_i \neq x_k$ with $x_i \in c_j$ and $X_i \subset D$ (resp. $\overline{x_i} \in c_j$ and $\overline{X_i} \subset D$), then $x_i = 1$ (resp. $x_i = 0$) and c_j is satisfied by A(D). So we can assume that, for any $x_i \neq x_k$ with $x_i \in c_j$ we have $X_i \not\subset D$. By Lemma 7, $\overline{X_i} \subset D$. And for any $x_i \neq x_k$ such that $\overline{x_i} \in c_j$ we have $\overline{X_i} \subset D$. So, by definition of A(D), we have $x_k = 1$. Since $x_k \in c_j$ (since W_i^k exists), c_j is satisfied by A(D).

Finally, assume that C_j is only dominated by \overline{W}_j^k in D, where $x_k = mv(D)$. If there exists $x_i \neq x_k$ such that $x_i \in c_j$ and $\overline{X_i} \subset D$ (resp. $\overline{x_i} \in c_j$ and $\overline{X_i} \subset D$), then $x_i = 1$ (respectively $x_i = 0$) so c_j is satisfied by A(D). Thus, by Lemma 7, we can assume that for any $x_i \neq x_k$ such that $x_i \in c_j$ (resp. $\overline{x_i} \in c_j$), we have $\overline{X_i} \subset D$ (resp. $X_i \subset D$). Let us show that there is no clause interval $C_{j'}$ dominated by a W_i^k interval of x_k in D and that satisfies, for any $x_i \neq x_k$, if $x_i \in c_{j'}$ then $\overline{X_i} \subset D$, and if $\overline{x_i} \in c_{j'}$ then $X_i \subset D$. This will imply $x_k = 0$ by construction and then the fact that c_j is satisfied.

Since D_s has no moving variable, there exists a dominating set before D in S' with no moving variable. Let D_r be the latest in S' amongst such dominating sets. By assumption, $mv(D_q) = x_k$ for any set D_q that comes earlier than D but later than D_r . Thus, by Claim 2, for any variable $x_i \neq x_k$, x_i has the same value in $A(D_r)$ and A(D).

Now, by assumption, for any $x_i \neq x_k$ with $x_i \in c_j$ (resp. $\overline{x_i} \in c_j$) we have $\overline{X_i} \subset D$ (resp. $X_i \subset D$). Thus, since x_i has the same value in D and D_r , if $x_i \in c_j$ (resp. $\overline{x_i} \in c_j$) then $\overline{X_i} \subset D_r$ (resp. $X_i \subset D_r$) and then, by Lemma 8, $W_j^i \notin D_r$ (resp. $\overline{W}_j^i \notin D_r$). Therefore, C_j is only dominated by \overline{W}_j^k in D_r . But since D_r has no moving variable, $\overline{X_k} \subset D_r$ by Lemma 7 and Lemma 8. Thus, by Lemma 8, for any $j' \neq j$, $W_{j'}^k \notin D_r$. So for any $j' \neq j$ such that $x_k \in c_{j'}$, $C_{j'}$ is dominated by at least one interval $W_{j'}^i$ or $\overline{W}_{j'}^i$ in D_r , where $x_i \neq x_k$. Lemma 8 ensures that if $C_{j'}$ is dominated by $W_{j'}^i$ (resp. $\overline{W}_{j'}^i$) in D_r then $X_i \subset D_r$ (resp. $\overline{X_i} \subset D_r$), and since x_i has the same value in D and D_r , it gives $X_i \subset D$ (resp. $\overline{X_i} \subset D$). Therefore, by Lemma 7, if a clause interval $C_{j'}$ is dominated by a W interval of x_k in D, then either there exists $x_i \neq x_k$ such that $x_i \in c_{j'}$ and $D(\overline{x_i}) \notin D$, or there exists $x_i \neq x_k$ such that $\overline{x_i} \in c'_j$ and $D(x_i) \notin D$. By definition of A(D), this implies that $x_k = 0$ in A(D). Since \overline{W}_j^k exists, $\overline{x_k} \in c_j$ thus c_j is satisfied by A(D).

Therefore, every clause of *F* is satisfied by A(D), which concludes the proof.

 \diamond

Claim 5. For any two dominating sets D_r and D_{r+1} of S', either $A(D_{r+1}) = A(D_r)$, or $A(D_{r+1})$ is reachable from $A(D_r)$ with a variable flip move.

Proof. By Claim 2, for any variable x_i such that $x_i \neq mv(D_r)$ and $x_i \neq mv(D_{r+1})$, x_i has the same value in $A(D_r)$ and $A(D_{r+1})$. Moreover, by definition of S', if both D_r and D_{r+1} have a moving variable then $mv(D_r) = mv(D_{r+1})$. Therefore, at most one variable change its value between $A(D_r)$ and $A(D_{r+1})$, which concludes the proof.

We now have all the ingredients to prove our main result:

of Theorem 2. Let $D_s = D_F(A_s)$ and $D_t = D_F(A_t)$. Lemma 9 and 11 ensure that (G_F, D_s, D_t) is a yes-instance of DSR_{TS} if and only if (F, A_s, A_t) is a yes-instance of SATR. Since SATR is PSPACE-complete [7], it gives the result.

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