k-colouring (m, n)-mixed graphs with switching

Richard C. Brewster*, Arnott Kidner†, Gary MacGillivray‡

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Abstract

A mixed graph is a set of vertices together with an edge set and an arc set. An (m, n)-mixed graph G is a mixed graph whose edges are each assigned one of m colours, and whose arcs are each assigned one of n colours. A *switch* at a vertex v of G permutes the edge colours, the arc colours, and the arc directions of edges and arcs incident with v. The group of all allowed switches is Γ.

Let $k \geq 1$ be a fixed integer and Γ a fixed permutation group. We consider the problem that takes as input an (m, n)-mixed graph G and asks if there a sequence of switches at vertices of G with respect to Γ so that the resulting (m, n)-mixed graph admits a homomorphism to an (m, n)-mixed graph on k vertices. Our main result establishes this problem can be solved in polynomial time for $k \leq 2$, and is NP-hard for $k \geq 3$. This provides a step towards a general dichotomy theorem for the Γ-switchable homomorphism decision problem.

1 Introduction

Homomorphisms of graphs (and in general relational systems) are well studied generalizations of vertex colourings [10]. Given a graph (or some generalization) G, the question of whether G admits a k-colouring, can be equivalently rephrased as "does G admit a homomorphism to a target on k vertices?".

In this paper we study homomorphisms of (m, n)-mixed graphs endowed with a switching operation under some fixed permutation group. (Formal definitions and precise statements of our results are given below.) Our main result is that the 2-colouring problem under these homomorphisms can be solved in polynomial time. As k-colouring for classical graphs can be encoded within our framework, k-colouring in our setting is NP-hard for fixed $k \geq 3$. That is, k-colouring for (m, n)-mixed graphs with a switching operation exhibits a dichotomy analogous to k-colouring of classical graphs [7]. Thus, our work maybe viewed as

*Department of Mathematics and Statistics, Thompson Rivers University, Kamloops, B.C., Canada

†Department of Mathematics and Statistics, University of Victoria, Victoria, B.C., Canada

‡Department of Mathematics and Statistics, University of Victoria, Victoria, B.C., Canada

a first step towards a dichotomy theorem for homomorphisms of (m,n)-mixed graphs with a switching operation. We remark that the k-colouring problem in our setting is not obviously a Constraint Satisfaction Problem [4, 6, 19] nor is membership in NP clear. These ideas are explored further in a companion paper [3].

We begin with the key definitions to state our main result. In this paper, all graphs and all groups are finite.

A *mixed graph* is a triple $G = (V(G), E(G), A(G))$ consisting of a set of vertices $V(G)$, a set of edges $E(G)$ of unordered pairs of vertices, and a set of arcs $A(G)$ of ordered pairs of vertices. Given pair of vertices u and v, there is at most one edge, or one arc, but not both, joining them. Further we assume G is loop-free. We will use uv to denote an edge or an arc with end points u and v where in the latter case the arc is oriented from u to v.

Mixed graphs were introduced by Nešetřil and Raspaud [15] as an attempt to unify the theories of homomorphisms of 2-edge coloured graphs and of oriented graphs. Numerous similarities between the two settings have been observed (see for example, [1, 12, 16]), whereas, Sen [17] provides examples highlighting key differences.

In this work we study edge and arc coloured generalizations of mixed graphs. Thus, our work may be viewed as a unification of homomorphisms of edge-coloured graphs and of arc-coloured graphs. Let m and n be non-negative integers. Denote by $[m]$ the set $\{1, 2, \ldots, m\}$. An *(m,n)-mixed graph* is a mixed graph $G = (V(G), E(G), A(G))$ together with functions $c : E(G) \to [m]$ and $d : A(G) \to [n]$ that assign to each edge one of m colours, and to each arc one of n colours respectively. (The colour sets for edges and arcs are disjoint.) The *underlying mixed graph* of G is $(V(G), E(G), A(G))$, i.e., the mixed graph obtained by ignoring edge and arc colours. The *underlying graph* of G is the graph obtained by ignoring edge and arc colours and arc directions. An (m,n)-mixed graph is a cycle if its underlying graph is a cycle and similarly for other standard graph theoretic terms such as path, tree, bipartite, etc.

Fundamental to our work is the following definition. An (m,n)-mixed graph is *monochromatic of colour* i if either every edge is colour i and there are no arcs, or every arc is colour i and there are no edges. While a monochromatic mixed graph with only edges is naturally isomorphic to its underlying graph, we note that we still view the edges as having colour i.

Let G and H be (m,n)-mixed graphs. A *homomorphism* of G to H is a function $h : V(G) \to V(H)$ such that if uv is an edge of colour i in G, then $h(u)h(v)$ is an edge of colour i of H, and if uv is an arc of colour j in G, then $h(u)h(v)$ is an arc of colour j in H. We denote the existence of a homomorphism of G to H by $G \to H$ or $h : G \to H$ when the name of the function is required.

We now turn our attention to the concept of switching an (m,n)-mixed graph at a vertex v. This generalizes the concept of switching edge colours or signs [2, 18] (permuting the colour of edges incident at v) and pushing digraphs [11] (reversing the direction of arcs incident at v). Let $\Gamma \leq S_m \times S_n \times S_2^n$ be a permutation group. An element of Γ will act on edge colours, arc colours, and arc directions. Specifically, the element is an ordered $(n+2)$-tuple $\pi =$

$(\alpha, \beta, \gamma_1, \gamma_2, \ldots, \gamma_n)$ where α acts on the edge colours, β acts on the arc colours, and γ_i acts on the arc direction of arcs of colour i. For the remainder of the paper, Γ will be a permutation group as described here.

Let G be a (m, n)-mixed graph, and $\pi = (\alpha, \beta, \gamma_1, \gamma_2, \ldots, \gamma_n) \in \Gamma$. Define $G^{(v,\pi)}$ as the (m, n)-mixed graph arising from G by *switching at vertex v with respect to π* as follows. Replace each edge vw of colour i by an edge vw of colour $\alpha(i)$. Replace each arc a of colour i incident at v (i.e., $a = vx$ or $a = xv$) with an arc of colour $\beta(i)$ and orientation $\gamma_i(a)$. Note, $\gamma_i(a) \in \{vx, xv\}$.

Given a sequence of ordered pairs from $V(G) \times \Gamma$, say $\Sigma = (v_1, \pi_1)(v_2, \pi_2) \ldots (v_k, \pi_k)$, we define *switching G with respect to the sequence Σ* as follows:

$$G^{\Sigma} = (G)^{(v_1,\pi_1)(v_2,\pi_2)\ldots(v_k,\pi_k)} = (G^{(v_1,\pi_1)})^{(v_2,\pi_2)(v_3,\pi_3)\ldots(v_k,\pi_k)}.$$

Note if we let $\Sigma^{-1} = (v_k, \pi_k^{-1}) \ldots (v_1, \pi_1^{-1})$, then $G^{\Sigma\Sigma^{-1}} = G^{\Sigma^{-1}\Sigma} = G$.

Given a subset of vertices, $X \subseteq V(G)$, we can switch at each vertex of X with respect to a permutation $\pi \in \Gamma$, the result of which we denote by $G^{(X,\pi)}$. This operation is well defined independently of the order in which we switch. If uv is an edge or arc with one end in X, say u, then we simply switch at u with respect to π. Suppose both ends of uv are in X. If uv is an edge of colour i, then after switching at each vertex of X, the edge will have colour $\alpha^2(i)$. If uv is an arc, then after switching the colour will be $\beta^2(i)$ and the direction will be $\gamma_{\beta(i)}\gamma_i(uv)$.

Two (m, n)-mixed graphs G and G' with the same underlying graph are Γ-*switch equivalent* if there exists a sequence of switches Σ such that $G^{\Sigma} = G'$. We may simply say *switch equivalent* when Γ is clear from context. Note since $V(G) = V(G')$, we are viewing both (m, n)-mixed graphs as labelled and thus are not considering equivalence under switching followed by an automorphism. Such an extension of equivalence is possible but unnecessary in this work. Since Γ is a group, the following proposition is immediate.

Proposition 1.1. *Γ-switch equivalence is an equivalence relation on the set of (labelled) (m, n)-mixed graphs.*

We are now ready to define switching homomorphisms. Our definition naturally builds on homomorphisms of signed graphs [8, 14] and push homomorphisms of digraphs [11]. Let G and H be (m, n)-mixed graphs. A *Γ-switchable homomorphism* of G to H is a sequence of switches Σ together with a homomorphism $G^{\Sigma} \to H$. We denote the existence of such a homomorphism by $G \to_{\Gamma} H$, or $f : G \to_{\Gamma} H$ when we wish to name the mapping. Observe the notation $G \to H$ refers to a homomorphism of (m, n)-mixed graphs without switching, and $G \to_{\Gamma} H$ refers to switching G followed by a homomorphism of (the resulting) (m, n)-mixed graphs.

A useful fact is the following. If $G \to_{\Gamma} H$, then $G \to_{\Gamma} H^{(v,\pi)}$ for any $v \in V(H)$ and any $\pi \in \Gamma$. To see this let Σ be a sequence of switches such that $f : G^{\Sigma} \to H$. Let $X = f^{-1}(v) \subseteq V(G^{\Sigma})$. It is easy to see the same vertex mapping $f : V(G) \to V(H)$ defines a homomorphism $(G^{\Sigma})^{(X,\pi)} \to H^{(v,\pi)}$. As a result of this observation, we have two immediate corollaries. First, Γ-switchable homomorphisms compose. Second, when studying the question "does G admit a Γ-switchable homomorphism to H?" we are free to replace H with any H' switch equivalent to H.

For (classical) graphs, G is k-colourable if and only if it admits a homomorphism to a graph H of order k. Analogously, we say an (m, n)-mixed graph G is Γ-*switchable* k-*colourable*, if there is an (m, n)-mixed graph H of order k such that $G \to_\Gamma H$. The corresponding decision problem is defined as follows. Let $k \geq 1$ be a fixed integer and $\Gamma \leq S_m \times S_n \times S_2^n$ be a fixed group. We define the following decision problem.

> Γ-SWITCHABLE k-COL
> INPUT: An (m, n)-mixed graph G.
> QUESTION: Is G Γ-switchable k-colourable?

Our main result is the following dichotomy result for Γ-SWITCHABLE k-COL.

Theorem 1.2. *Let $k \geq 1$ be an integer and $\Gamma \leq S_m \times S_n \times S_2^n$ be a group. If $k \leq 2$, then* Γ-SWITCHABLE k-COL *is solvable in polynomial time. If $k \geq 3$, then* Γ-SWITCHABLE k-COL *is NP-hard.*

The NP-hardness half of the dichotomy is immediate.

Proposition 1.3. *For $k \geq 3$,* Γ-SWITCHABLE k-COL *is NP-hard.*

Proof. Let G be an instance of k-colouring (for classical graphs). Let G' be the (m, n)-mixed graph obtained from G by assigning each edge colour 1. If G is k-colourable, then clearly G' is k-colourable. (Assign all edges in G' and K_k the colour 1 and use the same mapping.) Conversely, if G' is k-colourable, then the Γ-switchable homomorphism induces a homomorphism of the underlying graphs showing G is k-colourable. □

For an Abelian group we remark that if G and G' are switch equivalent, then there is a sequence of switches Σ of length at most $|V(G)|$ so that $G^\Sigma = G'$. (This is discussed in more detail below.) Thus when Γ is Abelian, Γ-SWITCHABLE k-COL is in NP, and we can conclude for $k \geq 3$, the problem is NP-complete. The situation for non-Abelian groups is more complicated and is studied further in [3].

It is trivial to decide if an (m, n)-mixed graph is 1-colourable. Thus to complete the proof we settle the case $k = 2$. Results are known when Γ belongs to certain families of groups [5, 13]. The remainder of the paper establishes the problem is polynomial time solvable for all groups Γ.

We conclude the introduction with a remark on the general homomorphism problem. Let H be a fixed (m, n)-mixed graph and Γ a fixed permutation group.

> Γ-HOM-H
> INPUT: An (m, n)-mixed graph G.
> QUESTION: Does G admit a Γ-switchable homomorphism to H?

The complexity of Γ-HOM-H has been investigated for the same families of groups as Γ-switchable k-colouring in [5, 13]. The following theorem is an immediate corollary to our main result.

Theorem 1.4. *Let H be a 2-colourable (m, n)-mixed graph, then Γ-*Hom*-H is polynomial time solvable.*

2 Restriction to m-edge coloured graphs

If a non-trivial (m, n)-mixed graph G is 2-colourable, then the target of order 2 to which G maps must be a monochromatic K_2 or a monochromatic tournament T_2. In the former case G must have only edges and in the latter only arcs. Moreover, the underlying graph of G must be bipartite as a 2-colouring of G induces a 2-colouring of the underlying graph.

In this section we focus on the case where G has only edges and is bipartite. For ease of notation, and to align with the existing literature, we will refer to G as an *m-edge coloured graph.* Recall we use $[m]$ as the set of edge colours, and in this case we may restrict Γ to be a subgroup of S_m. We let H be the m-edge coloured K_2 with its single edge of colour i, and denote H by K_2^i.

We begin with some key observations. Let G be an m-edge coloured graph. If $G \to_\Gamma K_2^i$, then every colour appearing on an edge of G must belong to the orbit of i under Γ; otherwise, G is a no instance. Therefore, we make the assumption that Γ acts transitively on $[m]$. Under this assumption K_2^i is switch equivalent to K_2^j for any $j \in [m]$. Thus we have the following proposition.

Proposition 2.1. *Fix $i \in [m]$. Let G be a bipartite m-edge coloured graph. The following are equivalent.*

(1) $G \to_\Gamma K_2^i$,

(2) $G \to_\Gamma K_2^j$ for any $j \in [m]$,

(3) G can be switched to be monochromatic of some colour j.

Proof. The implication (1) $\Rightarrow$ (2) follows from the fact that $K_2^i \to_\Gamma K_2^j$ for any $j \in [m]$ by the transitivity assumption. The implication (2) $\Rightarrow$ (3) is trivial. Suppose G can be switched to be monochromatic of some colour j. Let G have the bipartition $X \cup Y$. Since Γ is transitive, there is $\pi \in \Gamma$ such that $\pi(j) = i$. Then $G^{(X,\pi)}$ is monochromatic of colour i implying $G \to_\Gamma K_2^i$. □

We have reduced the problem of determining whether an m-edge coloured graph G is 2-colourable to testing if G is bipartite and can be switched to be monochromatic of some colour j.

In the case of signed graphs (2-edge colours), G can be switched to be monochromatic of colour j if and only if each cycle of G can be switched to be a monochromatic cycle of colour j [18]. We shall show the same result holds for bipartite m-edge coloured graphs. However, for our setting the question of when a cycle can be switched to be monochromatic is more complicated. Hence, we begin by characterizing when an m-edge coloured even cycle can be made monochromatic. To this end, let G be a m-edge coloured cycle of length $2k$ on vertices $v_0, v_1, \ldots, v_{2k-1}, v_0$. By switching at v_1, the edge v_0v_1 can be made colour i. Next by switching at v_2, the edge v_1v_2 can be made colour i. Continuing, we see that G can be switched so that all edges except $v_{2k-1}v_0$ are colour i. For $i, j \in [m]$, we say the cycle G is *nearly monochromatic of colours* (i, j) if G has $2k-1$ edges of colour i and 1 edge of colour j. Thus the problem of determining if an even cycle can be switched to be monochromatic is reduced to the problem of determining if a nearly monochromatic cycle of length $2k$ can be switched to be monochromatic.

Let G be a cycle of length $2k$ that is nearly monochromatic of colours (i, j). We define a relation on $[m]$ by $j \sim_{2k} i$ if G is Γ-switchably equivalent to a monochromatic C_{2k} of colour i or equivalently $G \to_\Gamma K_2^i$.

As the definition suggests, the relation is an equivalence relation.

Lemma 2.2. *The relation $\sim_{2k}$ is an equivalence relation.*

Proof. The relation is trivially reflexive.

To see $\sim_{2k}$ is symmetric, assume $j \sim_{2k} i$. Let G be a cycle of length $2k$ that is nearly monochromatic of colour (j, i). Label the vertices of the cycle in the natural order as $v_0, v_1, \ldots, v_{2k-1}, v_0$ where v_0v_{2k-1} is the unique edge of colour i. Suppose $\pi(j) = i$. Let $\Sigma = (v_1, \pi), (v_3, \pi), \ldots, (v_{2k-3}, \pi)$. Then G^Σ is nearly monochromatic of colour (i, j), with edge $v_{2k-2}v_{2k-1}$ being the unique edge of colour j. By assumption there is a sequence of switches, say Σ', so that $G^{\Sigma\Sigma'}$ is monochromatic of colour i, giving $G \to_\Gamma K_2^i$. Thus, $G \to_\Gamma K_2^j$ by Proposition 2.1. That is, G can be made monochromatic of colour j or $i \sim_{2k} j$.

To prove $\sim_{2k}$ is transitive, suppose $i \sim_{2k} j$ and $j \sim_{2k} l$. Let G, G', and G'' be m-edge coloured cycles of length $2k$ each with the vertices $v_0, v_1, \ldots, v_{2k-1}$. (Technically, we are considering three distinct edge colourings of the same underlying graph.) Suppose G, G', and G'' are nearly monochromatic of colours (j, i), (l, j), and (l, i) respectively. There are $2k-1$ edges of colour j in G with edge v_0v_{2k-1} of colour i in G. Similarly there are $2k-1$ edges of colour l in G' with edge v_0v_{2k-1} of colour j in G' and $2k-1$ edges of colour l with edge v_0v_{2k-1} of colour i in G''. We shall show G'' can be switched to be monochromatic of colour l.

By hypothesis, there is a sequence Σ' such that $G'^{\Sigma'}$ is monochromatic of colour l. In particular, under Σ' all edges of colour l remain colour l, and the edge v_0v_{2k-1} changes from j to l. Thus, if we apply Σ' to G'' the edges of colour l remain colour l and the product of those switches at v_0 and v_{2k-1} changes v_0v_{2k-1} from colour i to colour $\sigma(i)$ for some $\sigma \in \Gamma$. We observe by the fact that $G'^{\Sigma'}$ is monochromatic, $\sigma(j) = l$.

We now construct a modified inverse of Σ'. Let Σ'' be the subsequence of Σ' consisting of the switches only at v_0 or v_{2k-1}. That is, Σ'' is a subsequence $(v_{s_0}, \pi_0), (v_{s_1}, \pi_1), \ldots, (v_{s_t}, \pi_t)$ where each $v_{s_r} \in \{v_0, v_{2k-1}\}$. Let X (respectively Y) be the vertices of G'' with even (respectively odd) subscripts. Starting with $G''^{\Sigma'}$ apply the following sequence of switches. For $r = t, t-1, \ldots, 0$, if $v_{s_r} = v_0$, then apply the switch (X, π_r^{-1}); otherwise, $v_{s_r} = v_{2k-1}$ and apply the switch (Y, π_r^{-1}). The net effect is to apply σ^{-1} to each edge of $G''^{\Sigma'}$. Thus

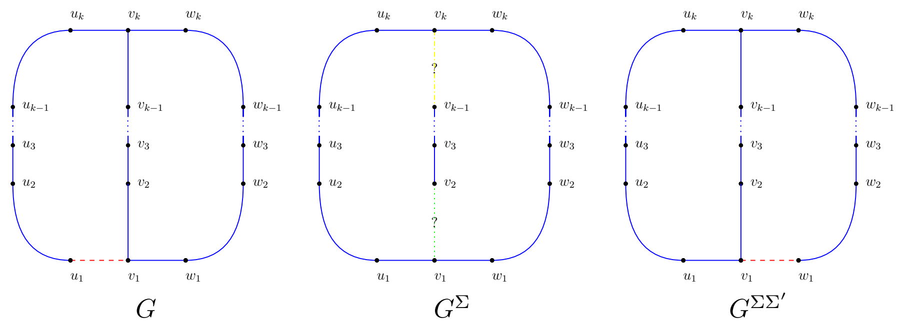

Figure 1: Switching of the theta graph in Theorem 2.3. Solid blue edges are colour i and dashed red edges are colour j.

each edge of colour l switches to j and the edge v_0v_{2k-1} of colour $\sigma(i)$ becomes colour i. That is, we can switch G'' to be G. By hypothesis, G can be switched to be monochromatic of colour j. By Proposition 2.1, the resulting m-edge coloured graph can be switched to be monochromatic of colour l, i.e., $i \sim_{2k} l$, as required. □

We denote the equivalence classes with respect to $\sim_{2k}$ by $[i]_\Gamma^{2k} = \{j|j \sim_{2k} i\}$. We now show that these classes are independent of cycle length (for even length cycles).

Theorem 2.3. *Let $\Gamma \leq S_m$ and $i \in [m]$. Then $[i]_\Gamma^{2l} = [i]_\Gamma^{2k}$ for all $l, k \in \{2, 3, \dots\}$.*

Proof. Let $i \in [m]$ and let k be an integer $k \geq 2$. We show $[i]_\Gamma^4 = [i]_\Gamma^{2k}$ from which the result follows.

Suppose $j \in [i]_\Gamma^4$. Let G be a cycle of length $2k$ and H a cycle of length 4 where both are nearly monochromatic of colours (i, j). Since $G \to H$ and by hypothesis, $H \to_\Gamma K_2^i$, we have $G \to_\Gamma K_2^i$ and thus $j \in [i]_\Gamma^{2k}$.

Conversely, suppose $j \in [i]_\Gamma^{2k+2}$. We will show $j \in [i]_\Gamma^{2k}$ from which we can conclude by induction that $j \in [i]_\Gamma^4$. Let G be the m-edge coloured graph constructed as follows. Let $v_1, v_2, \dots, v_k$; $u_1, u_2, \dots, u_k$; and $w_1, w_2, \dots, w_k$ be three disjoint paths of length $k-1$. Join v_1 to both u_1 and w_1, and v_k to both u_k and w_k. Each edge is colour i with the exception of v_1u_1 which is colour j. (Thus, G is the θ-graph with path lengths $k+1, k-1, k+1$.) Denoted the cycles $u_1, \dots, u_k, v_k, \dots, v_1, u_1$ and $w_1, \dots, w_k, v_k, \dots, v_1, w_1$ by C_1 and C_2 respectively. Observe both have length $2k$, C_1 is nearly monochromatic of colours (i, j) and C_2 is monochromatic of colour i. Finally, let C_3 be the cycle $u_1, \dots, u_k, v_k, w_k, \dots, w_1, v_1, u_1$. The cycle C_3 has length $2k+2$ and is nearly monochromatic of colours (i, j). See Figure 1.

By assumption there exists a sequence of switches Σ (acting on the vertices of C_3) such that in G^{Σ} the cycle C_3 is monochromatic of colour i. We note that

v_1v_2 and $v_{k-1}v_k$ might not be of colour i in G^{Σ}.

There is an automorphism φ of the underlying graph G that fixes each v_l, $l = 1, 2, \ldots, k$, and interchanges each u_l with w_l. We apply Σ^{-1} to $\varphi(G^{\Sigma})$ as follows. Let Σ' be the sequence obtained from Σ by reversing the order of the sequence, replacing each permutation with its inverse permutation and replacing all switches on vertices u_l with switches on w_l and vice versa. (Switches on v_1 and v_k are applied to v_1 and v_k respectively.) Then in $G^{\Sigma\Sigma'}$ we see that C_1 is monochromatic of colour i. Therefore $[i]_{\Gamma}^{2k} \supseteq [i]_{\Gamma}^{2k+2}$ for all $k \geq 2$. We conclude $[i]_{\Gamma}^{4} = [i]_{\Gamma}^{2k}$ for all $k \geq 2$. □

As the equivalence classes depends only on the group and not the length of the cycle, we henceforth denote these classes as $[i]_{\Gamma}$. If $j \in [i]_{\Gamma}$, we say i can be Γ-*substituted* for j; that is, the single edge of colour j in the cycle can be switched to colour i. We call $[i]_{\Gamma}$ the Γ-*substitution class* for i.

For a fixed m and Γ, $[i]_{\Gamma}$ can be computed in constant time as there is a constant number of m-edge coloured 4-cycles, and a constant number of (single) switches that can be applied to these cycles, from which the equivalence classes can be computed using the transitive closure.

Theorem 2.4. *Let G be an m-edge coloured C_{2k}. It can be determined in polynomial time whether there is a Γ-switchable homomorphism of G to K_2^i.*

Proof. As described above, we can switch G to be nearly monochromatic of colours (i, j), for some j. Then $G \to_{\Gamma} K_2^i$ if and only if $j \in [i]_{\Gamma}$. Testing this condition can be done in constant time. □

We now show the Γ-Hom-K_2^i problem is polynomial time solvable. This is accomplished by showing the problem of determining whether a given m-edge coloured bipartite graph can be made monochromatic of colour i is polynomial time solvable.

We begin with the following observation that trees can always be made monochromatic.

Lemma 2.5. *Let T be a m-edge coloured tree, then for any Γ, $T \to_{\Gamma} K_2^i$.*

Proof. Let T be a m-edge coloured tree. Let $v_1, v_2, \ldots, v_{|T|}$ be a depth first search ordering of T rooted at v_1. For each $k \in 2, \ldots, |T|$, switch at v_k so that the edge from v_k to its parent in the depth first search ordering has colour i. We observe that if the subtree $T[v_1, \ldots, v_{k-1}]$ is monochromatic of colour i, then after switching at v_k, so is the subtree $T[v_1, \ldots, v_k]$. □

Let G and H be m-edge coloured graphs such that H is a subgraph of G. A *retraction* from G to H, is a homomorphism $r : G \to H$ such that $r(x) = x$ for all $x \in V(H)$. We shall use the following result of Hell [9].

Theorem 2.6. *Let G be a bipartite graph. Suppose P is a shortest path from u to v in G. Then G admits a retraction to P.*

We now show, for general m-edge coloured graphs G, testing if $G \to_\Gamma K_2^i$ comes down to testing if each cycle admits a Γ-switchable homomorphism to K_2^i. To this end define $\mathcal{C}(G)$ to be the set of cycles in an m-edge coloured graphs G, and $\mathcal{F}_\Gamma$ to be the collection of cycles C such that $C \not\to_\Gamma K_2^i$.

Theorem 2.7. *Let G be a connected m-edge coloured graph and Γ a transitive group acting on $[m]$. Suppose $i \in [m]$. The following are equivalent.*

(1) $G \to_\Gamma K_2^i$.

(2) For all cycles $C \in \mathcal{C}(G)$, $C \to_\Gamma K_2^i$.

(3) G is bipartite and for any spanning T of G, there is a switching sequence Σ such that in G^Σ, T is monochromatic of colour i and for each cotree edge the colour i can be Γ-substituted for the colour of the cotree edge.

(4) For all cycles $C \in \mathcal{F}_\Gamma$, $C \not\to_\Gamma G$

Proof. We first prove the equivalence of statements (1), (2), and (3).

(1) $\Rightarrow$ (2) is trivially true.

(2) $\Rightarrow$ (3). We first observe that G must be bipartite as all cycles in the underlying graph map to K_2. Let T be a spanning tree in G and let Σ be the switching sequence constructed as in the proof of Lemma 2.5. Then T is monochromatic of colour i in G^Σ. Let e be a cotree edge of colour j. The fundamental cycle C_e in $T + e$ is nearly monochromatic of colours (i, j). By hypothesis $C \to_\Gamma K_2^i$. Hence, i Γ-substitutes for j.

(3) $\Rightarrow$ (1). As above, let T be a spanning tree that is monochromatic of colour i in G^Σ. Let $e_1, e_2, \ldots, e_k$ be an enumeration of the cotree edges of T. By hypothesis for each cotree edge e_t, its colour, say j (in G^Σ), belongs to $[i]_\Gamma$.

Let $T + \{e_1, \ldots, e_t\}$ be the subgraph of G^Σ induced by the edges $E(T) \cup \{e_1, \ldots, e_t\}$. Clearly $T \to_\Gamma K_2^i$. Suppose $T + \{e_1, \ldots, e_{t-1}\} \to_\Gamma K_2^i$. Let $e_t = uv$ have colour j. Let P be a shortest path from u to v in $T + \{e_1, \ldots, e_{t-1}\}$. By [9], there is a retraction $r : T + \{e_1, \ldots, e_{t-1}\} \to P$ with $r(u) = u$ and $r(v) = v$. Adding the edge e_t shows $T + \{e_1, \ldots, e_t\} \to_\Gamma P + e_t$ where $P + e_t$ is a nearly monochromatic cycle of colours (i, j). By assumption i Γ-substitutes for j, so $P + e_t \to_\Gamma K_2^i$ and by composition $T + \{e_1, \ldots, e_t\} \to_\Gamma K_2^i$. By induction, $G \to_\Gamma K_2^i$.

Finally, we show (1) and (4) are equivalent. If there is $C \in \mathcal{F}_\Gamma$ such that $C \to_\Gamma G$, then $G \not\to_\Gamma K_2^i$. Conversely, if $G \not\to_\Gamma K_2^i$, then by (2), there is a cycle C in G such that $C \not\to_\Gamma K_2^i$. In particular, $C \in \mathcal{F}_\Gamma$ and the inclusion map gives $C \to_\Gamma G$. $\square$

Given an m-edge coloured graph G, it is easy to test condition (3) for each component. Checking G is bipartite and the switching of a spanning forest can be done in linear time in $|E(G)|$. The look up for each cotree edge requires constant time.

However, the theorem actually gives us a certifying algorithm which we now outline (under the assumption G is connected). First test if G is bipartite. If it is not, then we discover an odd cycle certifying a no instance. Otherwise construct a spanning tree, and switch so that the tree is monochromatic of colour i. Either the colour of each cotree edge belongs to $[i]_\Gamma$ or we discover a cotree edge that does not. In the latter case we have a cycle of $C \in \mathcal{F}_\Gamma$ that certifies G is a no instance.

Thus assume all cotree edges have colours in $[i]_\Gamma$. The proof of Theorem 2.7 provides an algorithm for switching G to be monochromatic of colour i through lifting the switching of the retract $P+e_t$ to all of G. We show how using a similar idea with C_4 also works and gives a clearer bound on the running time. Let j be the colour of a cotree edge, say uv. Recall $j \in [i]_\Gamma^4$. Let H be a C_4 with vertices labelled as v_0, v_1, v_2, v_3 and edges coloured as v_0v_3 is colour j and all other edges are colour i. Let Σ be a switching sequence so that H^Σ is monochromatic of colour i. Let X (respectively Y) be the vertices of G in the same part of the bipartition as u (respectively v). For each (v_i, π_i) in Σ we apply the same switch π_i in G at u if $v_i = v_0$; at $X\backslash\{u\}$ if $v_i = v_2$; at v if $v_i = v_3$; and at $Y\backslash\{v\}$ if $v_i = v_1$. At the end of applying all switches in Σ, edges in G that were of colour i remain colour i, and the cotree edge uv switches from j to i. As $|\Sigma|$ is constant (in $|\Gamma|$), this switching sequence for uv requires $O(|V(G)|)$ switches. In this manner the concatenation of $|E(G)| - |V(G)| + 1$ such switching sequences (together with the switches required to make T monochromatic) switch G to be monochromatic of colour i. This sequence together with the bipartition of G certifies that $G \to_\Gamma K_2^i$. We have the following.

Corollary 2.8. *The problem* Γ-HOM-K_2^i *is polynomial time solvable by a certifying algorithm.*

3 General (m,n)-coloured graphs

In this section we show the Γ-2-COL problem is polynomial time solvable. As noted above, a general (m,n)-mixed graph G is 2-colourable if it only has edges and for some edge colour i, $G \to_\Gamma K_2^i$ or it only has arcs and for some arc colour i, $G \to_\Gamma T_2^i$. Having established the Γ-HOM-K_2^i problem is polynomial time solvable, we now show Γ-HOM-T_2^i polynomially reduces to Γ-HOM-K_2^i. This establishes the polynomial time result of Theorem 1.2 which we restate.

Theorem 3.1. *The* Γ-SWITCHABLE 2-COL *problem is polynomial time solvable.*

Proof. Let G be an instance of Γ-SWITCHABLE 2-COL, i.e., an (m,n)-mixed graph. If G is not bipartite, we can answer No. If G has both edges and arcs, then we can answer No. If G only has edges, then by Corollary 2.8 we can choose any edge colour i (we still assume Γ is transitive) and test $G \to_\Gamma K_2^i$ in polynomial time.

Thus assume G is bipartite with bipartition (A,B) and has only arcs. Analogous to Section 2, we can view Γ as acting transitively on the n-arc colours. If Γ does not allow any arc colours to switch direction, i.e., for all $\pi \in \Gamma$, $\gamma_i(uv) = uv$ for all i, then G must have all its arcs from say A to B; otherwise, we can say

No. At this point G may be viewed as an n-*edge* coloured graph. (We can ignore the fixed arc directions.) We apply the results of Section 2.

Finally, we may assume G is bipartite, with only arcs, and Γ acts transitively on arc colours and directions. That is, for any arc colours i and j, Γ contains a permutation π_1 (respectively π_2) that takes an arc uv of colour i to an arc uv (respectively vu) of colour j.

We now construct a $(2n)$-*edge* coloured graph G' as follows. Let $V(G') = V(G)$. If there is an arc of colour i from $u \in A$ to $v \in B$, we put an edge uv of colour i^+ in G', and if there is an arc of colour i from $v \in B$ to $u \in A$, we put an edge uv of colour i^- in G'.

From Γ we construct a new group $\Gamma' \leq S_{2n}$. Note that Γ as described above acts on (m, n)-mixed graphs and Γ' will be naturally restricted to acting on $(2n)$-edge coloured graphs. Let $\pi = (\alpha, \beta, \gamma_1, \ldots, \gamma_n) \in \Gamma$. Define $\pi' \in \Gamma'$ as follows. For each arc colour i,

$$\pi'(i^+) = \begin{cases} \beta(i)^+ & \text{if } \gamma_i(uv) = uv \\ \beta(i)^- & \text{if } \gamma_i(uv) = vu \end{cases} \quad \text{and} \quad \pi'(i^-) = \begin{cases} \beta(i)^- & \text{if } \gamma_i(uv) = uv \\ \beta(i)^+ & \text{if } \gamma_i(uv) = vu \end{cases}$$

It can be verified that the mapping $\pi \to \pi'$ is a group isomorphism.

The translation of G to G' can be expressed as a function $F(G) = G'$. It is straightforward to verify F is a bijection from n-arc coloured graphs to $2n$-edge coloured graphs provided we fix the bipartition $V(G) = A \cup B$. Moreover, if $\pi \in \Gamma$ and π' is the resulting permutation in Γ', then again it is easy to verify that $F(G^{(v,\pi)}) = (G')^{(v,\pi')}$ for any v in $V(G) = V(G')$.

Suppose $G \to_\Gamma T_2^i$. By the transitivity of Γ, we may assume that T_2^i has its tail in A, and thus all arcs in G can be switched to be colour i with their tail in A. The corresponding switches on G' switch all edges to colour i^+. That is, $G' \to_{\Gamma'} K_2^{i^+}$. On the other hand, if $G' \to_{\Gamma'} K_2^{i^+}$, then the corresponding switches on G show that $G \to_\Gamma T_2^i$ (with the vertices of A mapping to the tail of T_2^i). □

We conclude this section with a remark on the number of switches required to change the input G to be monochromatic. There are $|V(G)|-1$ switches required to change a spanning tree of G to be monochromatic of colour i. To change the cotree edges to colour i (assuming each is of a colour in $[i]_\Gamma$), we claim at most $c_\Gamma|V(G)|$ switches are required where c_Γ is a constant depending on Γ and the number of colours (m and n). We argue only for m-edge coloured graphs, given the reduction above. For (a labelled) C_4, there are m^4 edge colourings. For each vertex there are $|\Gamma|$ switches. The *reconfiguration graph* $\mathcal{C}$ has a vertex for each edge-colouring of C_4 and an edge joining two vertices is there is a single switch that changes one into the other. (The existence of inverses ensures this is an undirected graph.) Thus, $\mathcal{C}$ has order m^4 and is regular of degree $|\Gamma|$. Given $j \in [i]_\Gamma$, there is a path in $\mathcal{C}$ from a nearly monochromatic C_4 of colours (i, j) to a monochromatic C_4 of colour i. The switches on this path can be lifted to G so that the spanning tree remains of colour i and the cotree edge switches to colour i. The total number of switches is at most $\max\{\mathrm{diam}(\mathcal{C}')\} \cdot |V(G)|$ where $\mathcal{C}'$ runs over all components of $\mathcal{C}$. Thus we have the following.

Proposition 3.2. *Let G be a m-edge coloured bipartite graph. Let Γ be a group acting transitively on $[m]$. If G is Γ-switch equivalent to a monochromatic graph, then the sequence Σ of switches which transforms G to be monochromatic satisfies,*

$$|\Sigma| \leq |V(G)| - 1 + c_\Gamma |V(G)|(|E(G)| - |V(G)| + 1)$$

where c_Γ depends only on Γ and m.

In the case that Γ is abelian, the switches in Σ can be reordered, then combined, so that each vertex is switched only once.

4 Conclusion

We have established a dichotomy for the Γ-Switchable k-Col problem. This is a step in obtaining a dichotomy theorem for Γ-Hom-H for all (m,n)-mixed graphs H and all transitive permutation groups Γ. Work towards a general dichotomy is the focus of our companion paper [3].

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