**COT 6410 Spring 2022** **Midterm** **Name:____KEY__________**
**Raw Score ___ / 60** **Grade:_________**

*6* **1.** In each case below, consider **R1** to be Regular, **R2** to be finite, and **L1** and **L2** to be non-regular CFLs. Fill in the three columns with **Y** or **N**, indicating what kind of language **L** can be. No proofs are required. Read $\subseteq$ as "contained in and may equal."
Put **Y** in all that are possible and **N** in all that are not.

| Definition of L | Regular? | CFL, non-Regular? | Not even a CFL? |
|---|---|---|---|
| $L = L1 \cap L2$ | *Y* | *Y* | *Y* |
| $L = L1 - R2$ | *Y* | *Y* | *N* |
| $L = \Sigma^* - R1$ | *Y* | *N* | *N* |
| $L \subseteq R1$ | *Y* | *Y* | *Y* |

*3* **2.** Choosing from among **(D) decidable**, **(U) undecidable**, categorize each of the following decision problems. No proofs are required. **L** is a language over **Σ**.

| Problem / Language Class | Regular | Context Free | Context Sensitive | Phrase Structured |
|---|---|---|---|---|
| L contains Σ ? | *D* | *D* | *D* | *U* |
| \|L\| is infinite ? | *D* | *D* | *U* | *U* |

*4* **3.** Prove that any class of languages, ***C***, closed under union, concatenation, intersection with regular languages, homomorphism and substitution (e.g., the Context-Free Languages) is closed under **Cut with Regular Sets**, denoted by the operator |||, where $L \in C$, **R** is Regular, **L** and **R** are both over the alphabet **Σ**, and
$L|||R = \{ x_1 x_2 \dots x_k \mid k \geq 1, \forall i\ y_i \in R \text{ and } x_1 y_1 x_2 y_2 \dots x_k y_k \in L \text{ where each } x_i \in \Sigma^+ \}$.
You may assume substitution **f(a) = {a, a'}**, and homomorphisms **g(a) = a'** and **h(a) = a, h(a') = λ**. Here $a \in \Sigma$ and **a'** is a new character associated with each such $a \in \Sigma$.
You only need give me the definition of **L|||R** in an expression that obeys the above closure properties; you do not need to prove or even justify your expression.

**L|||R =** $h(f(L) \cap (\Sigma^+ g(R))^+)$

*4* **4.** Specify True (**T**) or False (**F**) for each statement.

| Statement | T or F |
|---|---|
| The Context Sensitive Languages are closed under union | *T* |
| The Post Correspondence Problem is undecidable if \|Σ\| = 2 | *T* |
| The function **Univ(f,x)** = $\varphi_f$**(x)** is primitive recursive | *F* |
| If **Halt** $\leq_m$ **P** then **P** must be RE | *F* |
| The **RE** sets are closed under complement | *F* |
| Myhill-Nerode proves that every regular language has a minimum state DFA | *T* |
| If **P** is solvable then Rice's Theorem cannot apply to P | *T* |
| The incorrect traces of a Turing Machine's Computations form a CFL | *T* |

4 **5.** Let $\mathbf{P} = \langle\langle x_1,x_2,\ldots,x_n\rangle, \langle y_1,y_2,\ldots,y_n\rangle\rangle$, $x_i,y_1 \in \Sigma^+$, $1 \le i \le n$ , be an arbitrary instance of **PCP**. We can use **PCP's undecidability** to show the undecidability of the problem to determine if the language associated with a **Context Sensitive Grammar** is non-empty. Present a grammar, **G**, associated with an arbitrary instance of PCP, **P**, such that $\mathcal{L}(G)$ is non-empty if and only if there is a solution to **P**. Define $\mathbf{G} = (\{S, T\} \cup \Sigma, \{*\}, R, S)$ where **R is** the set of rules (this is your job):

$S \rightarrow x_i\, S\, y_i^R \mid x_i\, T\, y_i^R$
$a\, T\, a \rightarrow *\, T\, *$
$a\, * \rightarrow *\, \alpha$
$*\, a \rightarrow \alpha\, *$
$T \rightarrow *$

12 **6.** Choosing from among **(REC) recursive**, **(RE) re non-recursive, (coRE) co-re non-recursive**, **(NRNC) non-re/non-co-re**, categorize each of the sets in a) through d). Justify your answer by showing some minimal quantification of some known recursive predicate.

**a.)** $D = \{ <f, x> \mid \varphi_f(x)$ **converges in at most 10 steps** $\}$

*STP(f, x, 10)* — *REC*

**b.)** $B = \{ f \mid \varphi_f$ **is not total** $\}$

$\exists x\ \forall t \sim STP(f, x, t)$ — *NRNC*

**c.)** $C = \{ f \mid$ **domain**$(\varphi_f)$ **is non-empty** $\}$

$\exists <x, t>\ STP(f, x, t)$ — *RE*

**d.)** $A = \{ f \mid\ |\text{range}(\varphi_f)| \le 1 \}$

$\forall <x,y,t>[STP(f,x,t) \,\&\, STP(f,y,t) \Rightarrow VALUE(f,x,t)=VALUE(f,y,t)]$ — *CO-RE*

2 **7.** Looking back at Question 6, which of these are candidates for using Rice's Theorem to show their unsolvability? Check all for which Rice Theorem might apply.

**a)** ____ **b)** *X* **c)** *X* **d)** *X*

**8.** We wish to prove that, if **S** and its complement **S'** are both be non-empty and recursively enumerable, then S is recursive (decidable). There are two approaches. The first is based on the fact that there are algorithms, $\mathbf{f_S}$ and $\mathbf{f_{S'}}$, that enumerate **S** and **S'**, respectively. The second is based on the fact that there are procedures (partial recursive functions), $\mathbf{g_S}$ and $\mathbf{g_{S'}}$, whose domains are **S** and **S'**, respectively.

*3* **a.**) Define a characteristic function for **S** based on the existence of $\mathbf{f_S}$ and $\mathbf{f_{S'}}$.

$\chi_S(x) =$ ***$f_S(\mu y\,[\, f_S(y) = x \,||\, f_{S'}(y) = x \,]) = x$***

*3* **b.**) Define a characteristic function for **S** based on the existence of $\mathbf{g_S}$ and $\mathbf{g_{S'}}$.

$\chi_S(x) =$ ***$STP(g_S, x, \mu t\,[\, STP(g_S, x, t) \,||\, STP(g_{S'}, x, t) \,])$***

*6* **9.** Let sets **A** be a **non-empty** recursive (decidable) set and let **B** be re non-recursive (undecidable). Consider $\mathbf{C = \{ z \mid z = min(x,y),}$ where $\mathbf{x \in A}$ and $\mathbf{y \in B}$ }.
For (a)-(c), either show sets **A** and **B** and the resulting set **C**, such that **C** has the specified property (argue convincingly that it has the correct property) or demonstrate (prove by construction) that this property cannot hold.

**a.** Can **C** be recursive? Circle ***Y*** or **N.**

***$A = \{ 0 \}$ $\quad B = K$***

***$C = A = \{ 0 \}$ which is recursive***

**b.** Can **C** be re non-recursive? Circle ***Y*** or **N.**

***$A = \{ 2x \mid x \in \aleph \}$ $\quad B = \{ 2x + 1 \mid\mid x \in K \}$***

***$C = A \cup B$ as they are disjoint sets and each elements un each set is the minimum of some pair with the other set. The membership problem for K reduces to that of C and vice versa.***
***Thus, C is re, non-recursive***

**c.** Can **C** be non-re? Circle **Y** or ***N*.**
You may assume $\mathbf{A = range(f_A), B = range(f_B),}$ for some algorithms $\mathbf{f_A, f_B}$.

***$f_{min(A,B)}(\langle x, y \rangle) = min(f_A(x), f_B(y))$***

***The range of $f_{min(A,B)}$ is then the set of minimums of the sets A and B.***
***This shows C is the range of some procedure and so is RE***

**10.** Define **PseudoFIB (PF) = { f | for some input x, f(x+2) = f(x+1)+f(x) }**.

*2* **a.)** Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of this set. (Hint: Look at **c.)** and **d.)** to get a clue as to what this must be.)

***$\exists \langle x, t \rangle$ [ STP(f, x+2, t) & STP(f, x+1, t) & STP(f, x, t) &***

***VALUE(f, x+2, t) = VALUE(f, x+2, t) + VALUE(f, x, t) ]***

*5* **b.)** Use Rice's Theorem to prove that **PF** is undecidable.

***First, PF is non-trivial as C0 $\in$ PF and C1 $\notin$ PF***

***Next, let f and g, be arbitrary function indices such that $\forall x$ f(x) = g(x)***

***f $\in$ PF $\Leftrightarrow$ $\underline{\exists x}$ [ f(x+2) = f(x)+f(x+1 ]) $\Leftrightarrow$***

***$\underline{\exists x}$ [ g(x+2) =gf(x)+g(x+1 ]) since $\forall x$ f(x) = g(x)***

***$\Leftrightarrow$ g $\in$ PF***

***By the Strong Form of Rice's Theorem, PF is thus shown to be undecidable***

*5* **c.)** Show that **K $\leq_m$ PF**, where **K = { f | f(f)$\downarrow$ }**.

***Let f be an arbitrary function index***

***Define $\underline{\forall x}$ $F_f$ (x) = f(f) – f(f)***

***f $\in$ K $\Leftrightarrow$ $\underline{\forall x}$ $F_f$ (x) = 0 $\Rightarrow$ $F_f$ $\in$ PF***

***f $\notin$ K $\Leftrightarrow$ $\underline{\forall x}$ $F_f$ (x) $\uparrow$ $\Rightarrow$ $F_f$ $\notin$ PF***

***Thus K $\leq_m$ PF as was to be shown***

*1* **d.)** From **a.)** through **c.)** what can you conclude about the complexity of **PF (Recursive, RE, RE-COMPLETE, CO-RE, CO-RE-COMPLETE, NON-RE/NON-CO-RE)**?

***RE-COMPLETE since PF is RE, K $\leq_m$ PF, and K is RE-COMPLETE***