

- 6 1. In each case below, consider **R1** and **R2** to be Regular and **L1** and **L2** to be non-regular CFLs. Fill in the three columns with **Y** or **N**, indicating what kind of language **L** can be. No proofs are required. Read \supseteq as “contains and may equal.” Put **Y** in all that are possible and **N** in all that are not.

Definition of L	Regular?	CFL, non-Regular?	Not even a CFL?
$L = L1 / R1$	Y	Y	N
$L = R1 - L1$	Y	Y	Y
$L = R1 \cap L1$	Y	Y	N
$L \supseteq R1$	Y	Y	Y

- 3 2. Choosing from among **(D) decidable**, **(U) undecidable**, **(?) unknown**, categorize each of the following decision problems. No proofs are required. **L** is a language over Σ ; **w** is a word in Σ^*

Problem / Language Class	Regular	Context Free	Context Sensitive	Phrase Structured
$w \in L ?$	D	D	D	U
L is infinite ?	D	D	U	U

- 4 3. Prove that any class of languages, **C**, closed under union, concatenation, intersection with regular languages, homomorphism and substitution (e.g., the Context-Free Languages) is closed under **Double Interior Loss with Regular Sets**, denoted by the operator $\|$, where $L \in C$, **R** is Regular, **L** and **R** are both over the alphabet Σ , and $L\|R = \{ uwy \mid \exists v, x \in R, \text{ such that } uvwxy \in L \}$. You may assume substitution $f(a) = \{a, a'\}$, and homomorphisms $g(a) = a'$ and $h(a) = a, h(a') = \lambda$. Here $a \in \Sigma$ and a' is a new character associated with each such $a \in \Sigma$. You only need give me the definition of $L\|R$ in an expression that obeys the above closure properties; you do not need to prove or even justify your expression.

$L\|R = \underline{h(f(L) \cap \Sigma^* g(R) \Sigma^* g(R) \Sigma^*)}$

- 4 4. Specify True (T) or False (F) for each statement.

Statement	T or F
Rice's Theorem demonstrates the undecidability of the Halting Problem	F
The Context Free Languages are closed under intersection	F
The Ambiguity problem for Context Free Languages is undecidable	T
The Quotient of two Context Free Languages is Context Sensitive	F
An algorithm exists to determine if a Context Free Language is Σ^*	F
Every RE set can be generated by a Phrase Structured Grammar	T
The set difference of two Context Free Languages is Context Sensitive	T
There is an algorithm to determine if $L = \emptyset$, for L a Context Sensitive Language	F

- 4 5. Let $P = \langle \langle x_1, x_2, \dots, x_n \rangle, \langle y_1, y_2, \dots, y_n \rangle \rangle$, $x_i, y_i \in \Sigma^+$, $1 \leq i \leq n$, be an arbitrary instance of PCP. We can use PCP's undecidability to show the undecidability of the problem to determine if a Context Free Grammar is ambiguous. Present the grammar, G , associated with an arbitrary instance of PCP, P , such that $\mathcal{L}(G)$ is ambiguous if and only if there is a solution to P .

Define $G = (\{ S, X, Y \}, \Sigma \cup \{ [i] \mid 1 \leq i \leq n \}, S, R)$, where R is the set of rules:

$$\begin{aligned} S &\rightarrow X \mid Y \\ X &\rightarrow x_i X [i] \mid X \rightarrow x_i [i] & 1 \leq i \leq n \\ Y &\rightarrow y_i Y [i] \mid Y \rightarrow y_i [i] & 1 \leq i \leq n \end{aligned}$$

- 12 6. Choosing from among (REC) recursive, (RE) re non-recursive, (coRE) co-re non-recursive, (NRNC) non-re/non-co-re, categorize each of the sets in a) through d). Justify your answer by showing some minimal quantification of some known recursive predicate.

a.) $\{ f \mid \text{domain}(f) = \text{range}(f) \}$

NRNC

Justification: $\forall \langle x, t \rangle \exists \langle y, s \rangle [STP(f, x, t) \Leftrightarrow (STP(f, y, s) \ \&\& \ VALUE(f, y, s) = x)]$

b.) $\{ \langle f, x \rangle \mid f(x) = x \}$

RE

Justification: $\exists t [STP(f, x, t) \ \&\& \ VALUE(f, x, t) = x]$

c.) $\{ f \mid f(x) \text{ converges in } x \text{ steps for at least one value of } x \}$

RE

Justification: $\exists x [STP(f, x, x)]$

d.) $\{ f \mid \text{whenever } f \text{ converges, } f(x) = x \}$

coRE

Justification: $\forall \langle x, t \rangle [STP(f, x, t) \Rightarrow (VALUE(f, x, t) = x)]$

- 2 7. Looking back at Question 6, which of these are candidates for using Rice's Theorem to show their unsolvability? Check all for which Rice Theorem might apply.

a) X

b) X

c) _____

d) X

- 6 8. Let S be an arbitrary, non-empty re/semi-decidable set. One definition is that S is the range of some total recursive f_s . Using f_s , show that S is the domain of some partial recursive function g_s . Here the function g_s that you define based on the existence of f_s semi-decides S .

$$g_s(x) = \exists y [f_s(y) = x]$$

Let S be an arbitrary, non-empty re/semi-decidable set. One definition is that S is the domain of some partial recursive function g_s . Using g_s and the fact that S is non-empty (you may assume c is some element guaranteed to be in S), show that S is the range of some total recursive f_s . Here the function f_s that you define based on the existence of g_s enumerates the elements of S . Hint: Each element of S is enumerated a countably infinite number of times by your function f_s .

$$f_s(\langle x, t \rangle) = x * STP(g_s, x, t) + c * (1 - STP(g_s, x, t))$$

- 6 9. Show example sets A and B , where A is non-empty and recursive, and B is re non-recursive and
- a.) $\text{Max}(A, B) = \{z \mid z = \max(x, y) \text{ where } x \in A \text{ and } y \in B\}$ is recursive

$$A = \mathcal{N}, B = K, \text{Max}(A, B) = \mathcal{N} - \{x \mid x < \text{least value in } K\} = \{x \mid x \geq \text{least value in } K\}$$

- b.) $\text{Max}(A, B) = \{z \mid z = \max(x, y) \text{ where } x \in A \text{ and } y \in B\}$ is re non-recursive

$$A = \{0\}, B = K, \text{Max}(A, B) = K$$

Hint: Consider $B = K = \{f \mid \varphi_f(f) \downarrow\}$

Note: You must specify the results of $\text{Max}(A, B)$ for each case above.

10. Define **SuccessorLike (SL)** = $\{ f \mid \text{for some input } x, f(x) = x+1 \}$.

- 2 a.) Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of this set. (Hint: Look at **c.**) and **d.**) to get a clue as to what this must be.)

$$\exists \langle x, t \rangle [STP(f, x, t) \ \&\& \ (VALUE(f, x, t) = x+1)]$$

- 5 b.) Use Rice's Theorem to prove that **SL** is undecidable.

$$S(x) = x+1 \in SL ; I(x) = x \notin SL \quad // \text{SL is non-trivial}$$

Let f and g be two arbitrary function indices such that $\forall x [f(x) = g(x)]$.

$$f \in SL \Leftrightarrow \exists x f(x) = x+1 \Rightarrow \text{for some } x_0, f(x_0) = x_0+1 \Rightarrow g(x_0) = x_0+1 \Rightarrow \exists x g(x) = x+1 \Rightarrow g \in SL$$

$$f \notin SL \Leftrightarrow \text{for no } x \text{ does } f(x) = x+1 \Leftrightarrow \text{for no } x \text{ does } g(x) = x+1 \Leftrightarrow g \notin SL // \text{Can just do this one}$$

- 4 c.) Show that $\mathbf{K} \leq_m \mathbf{SL}$, where $\mathbf{K} = \{ f \mid f(f) \downarrow \}$.

Let f be an arbitrary function index. Define $\forall x F_f(x) = f(f) - f(x) + x + 1$

$$f \in \mathbf{K} \Leftrightarrow f(f) \downarrow \Leftrightarrow \forall x F_f(x) = x + 1 \Rightarrow F_f \in SL$$

$$f \notin \mathbf{K} \Leftrightarrow f(f) \uparrow \Leftrightarrow \forall x F_f(x) \uparrow \Rightarrow F_f \notin SL$$

- 4 d.) Show that $\mathbf{SL} \leq_m \mathbf{K}$.

Let f be an arbitrary function index.

Define $\forall y F_f(y) = \exists \langle x, t \rangle [STP(f, x, t) \ \&\& \ (VALUE(f, x, t) = x+1)]$

$$f \in SL \Leftrightarrow \exists \langle x, t \rangle [STP(f, x, t) \ \&\& \ (VALUE(f, x, t) = x+1)] \Leftrightarrow \forall y F_f(y) \downarrow \Rightarrow F_f \in \mathbf{K}$$

$$f \notin SL \Leftrightarrow \sim \exists \langle x, t \rangle [STP(f, x, t) \ \&\& \ (VALUE(f, x, t) = x+1)] \Leftrightarrow \forall y F_f(y) \uparrow \Rightarrow F_f \notin \mathbf{K}$$

- 1 e.) From a.) through d.) what can you conclude about the complexity of **SL** (**Recursive**, **RE**, **RE-COMplete**, **CO-RE**, **CO-RE-COMplete**, **NON-RE/NON-CO-RE**)?

RE-COMplete