Assignment#4 Sample Key

1. Show S inf. rec. iff S is the range of a monotonically increasing function

- Let $\mathbf{f_S(x+1) > f_S(x)}$, and $\mathbf{Range(f_S(x)) = S}$. **S** is decided by the characteristic function
  $\chi_S(x) = \exists\ y \le x\ [\ \mathbf{f_S(y) == x}\ ]$
  The above works as **x** must show up within the first **x+1** numbers listed since $\mathbf{f_S}$ is monotonically increasing.
- Let **S** be infinite recursive. As **S** is recursive, it has a characteristic function where $\chi_S(x)$ is true iff **x** is in **S**.
  Define the monotonically increasing enumerating function $\mathbf{f_S(x)}$ where
  $\mathbf{f_S(0) = \mu\ x\ [}\ \chi_S(x)\ ]$
  $\mathbf{f_S(y+1) = \mu\ x{>}f_S(y)\ [}\ \chi_S(x)\ ]$
  As required, this enumerates the elements of **S** in order, low to high.

2. If S is infinite re, then S has an infinite recursive subset R

- Let f_S be an algorithm where **S = range(f_S)** is an infinite set
- Define the monotonically increasing function **$f_R(x)$** by
  $f_R(0) = f_S(0)$
  $f_R(y+1) = f_S(\ \mu\ x\ [\ f_S(x) > f_R(y)\ ]\ )$
- The above is monotonically increasing because each iteration seeks a larger number and it will always succeed since **S** is itself infinite and so has no largest value. Also, **R** is clearly a subset of **S** since each element is in the range of f_S.
- From #2, **R** is infinite recursive as it is the range of a monotonically increasing algorithm f_R.
- Combining, **R** is an infinite recursive subset of **S**, as was desired.

3. NotDominating(ND) = { f | for some x, f(x) < x }.

a.) Show some minimal quantification of some known primitive recursive predicate that provides an upper bound for the complexity of **ND**.

∃<x,t> [STP(f,x,t) & (VALUE(f,x,t) < x)]}

b.) Use Rice's Theorem to prove that **ND** is undecidable. Be Complete.

ND is non-trivial as **C0(x) = 0** ∈ **ND** and **S(x) = x+1** ∉ **ND**

Let f,g be two arbitrary indices of procedures such that ∀**x f(x) = g(x)**

f ∈ **ND** iff ∃**x f(x) < x iff f(x_0) < x_0 for some x_0 iff g(x_0) < x_0 as** ∀**x f(x) = g(x) implies** ∃**x g(x) < x iff g** ∈ **ND**

f ∉ **ND** iff ∀**x [f(x)↓ implies f(x)>x] iff** ∀**x [g(x)↓ implies g(x)>x] as** ∀**x f(x) = g(x) iff g** ∉ **ND**

c.) Show that **K = { f | f(f)** converges **}** is many-one reducible to **ND**.

Let **f** be an arbitrary index. From **f**, define ∀**x F_f(x) = f(f)-f(f)**.
f ∈ **K** implies ∀**x F_f(x) = 0** implies **F_f** ∈ **ND**.
f ∉ **K** implies ∀**x F_f(x)** diverges implies **F_f** ∉ **ND**.

Thus, **K $\leq_m$ ND**

d.) Show that **ND** is many-one reducible to **K = { f | f(f)** converges **}**

Let **f** be an arbitrary index. From **f**, define ∀**y F_f(y) =** ∃**<x,t> STP(f,x,t) & (VALUE(f,x,t) < x)**
f ∈ **ND** implies ∀**y F_f(y)** converges implies **F_f(F_f)** converges implies **F_f** ∈ **K**
f ∉ **ND** implies ∀**y F_f(y)** diverges implies **F_f** ∉ **K**.

Thus, **ND $\leq_m$ K**

4. AlwaysDominates(AD) = { f | for all x, f(x) > x }

a.) Show some minimal quantification of some known primitive recursive predicate that provides an upper bound for the complexity of **AD**.

∀x ∃t [STP(f,x,t) & (VALUE(f,x,t) > x)]}

b.) Use Rice's Theorem to prove that **AD** is undecidable. Be Complete.

AD is non-trivial as **S(x) = x+1** ∈ **AD** and **C0(x) = 0** ∉ **AD**

Let f,g be two arbitrary indices of procedures such that ∀x f(x) = g(x)

f ∈ **AD** iff **∀x f(x) < x iff ∀x g(x) < x iff g** ∈ **AD**

c.) Show that **TOT = { f |** for all **x**, f**(x)** converges **}** is many-one reducible to **AD**.

Let **f** be an arbitrary index. From **f**, define **∀x F_f(x) = f(x)-f(x) + x + 1**.
f ∈ **TOT** implies **∀x F_f(x) = x+1** implies **F_f** ∈ **AD.**
f ∉ **TOT** implies **∃x F_f(x)** diverges implies **F_f** ∉ **AD**.

Thus, **TOT $\leq_m$ AD**

d.) Show that **AD** is many-one reducible to **TOT = { f |** for all **x**, f**(x)** converges **}**

Let **f** be an arbitrary index. From **f**, define **∀x F_f(x) = μy [f(x) > x]**
f ∈ **AD** implies **∀x F_f(x)** converges implies **F_f** ∈ **TOT**
f ∉ **AD** implies **∃x F_f(x)** diverges implies **F_f** ∉ **TOT**.

Thus, **AD $\leq_m$ TOT**