Assignment#4 Sample Key
1. Show $S$ is the range of a monotonically increasing function

Let $f_S(x+1) > f_S(x)$, and $\text{Range}(f_S(x)) = S$. $S$ is decided by the characteristic function

$$\chi_S(x) = \exists y \leq x \ [ f_S(y) = x ]$$

The above works as $x$ must show up within the first $x+1$ numbers listed since $f_S$ is monotonically increasing.

Let $S$ be infinite recursive. As $S$ is recursive, it has a characteristic function where $\chi_S(x)$ is true iff $x$ is in $S$.

Define the monotonically increasing enumerating function $f_S(x)$ where

$$f_S(0) = \mu x \ [ \chi_S(x) ]$$
$$f_S(y+1) = \mu x > f_S(y) \ [ \chi_S(x) ]$$

As required, this enumerates the elements of $S$ in order, low to high.
2. If $S$ is infinite re, then $S$ has an infinite recursive subset $R$

• Let $f_S$ be an algorithm where $S = \text{range}(f_S)$ is an infinite set

• Define the monotonically increasing function $f_R(x)$ by

$$f_R(0) = f_S(0)$$
$$f_R(y+1) = f_S( \mu x \ [ f_S(x) > f_R(y) ] )$$

• The above is monotonically increasing because each iteration seeks a larger number and it will always succeed since $S$ is itself infinite and so has no largest value. Also, $R$ is clearly a subset of $S$ since each element is in the range of $f_S$.

• From #2, $R$ is infinite recursive as it is the range of a monotonically increasing algorithm $f_R$.

• Combining, $R$ is an infinite recursive subset of $S$, as was desired.
3. NotDominating(ND) = \{ f \mid \text{for some } x, f(x) < x \}.

a.) Show some minimal quantification of some known primitive recursive predicate that provides an upper bound for the complexity of ND.

\[ \exists <x,t> \ [\text{STP}(f,x,t) \land (\text{VALUE}(f,x,t) < x)] \]

b.) Use Rice's Theorem to prove that ND is undecidable. Be Complete.

ND is non-trivial as C0(x) = 0 \in ND and S(x) = x+1 \notin ND

Let f,g be two arbitrary indices of procedures such that \( \forall x f(x) = g(x) \)

f \in ND iff \( \exists x f(x) < x \) iff \( f(x_0) < x_0 \) for some \( x_0 \)

f \notin ND iff \( \forall x [f(x) \downarrow \text{implies } f(x) > x] \) iff \( \forall x [g(x) \downarrow \text{implies } g(x) > x] \)

f \in ND implies \( \exists x F_f(x) = 0 \) implies \( F_f \in K \)

f \notin ND implies \( \forall x F_f(x) \text{ diverges} \) implies \( F_f \notin K \).

Thus, K \leq_m ND

d.) Show that ND is many-one reducible to K = \{ f \mid f(f) \text{ converges} \}

Let f be an arbitrary index. From f, define \( \forall y F_f(y) = \exists <x,t> \ [\text{STP}(f,x,t) \land (\text{VALUE}(f,x,t) < x)] \)

f \in ND implies \( \forall y F_f(y) \text{ converges} \) implies \( F_f(F_f) \text{ converges} \) implies \( F_f \in K \)

f \notin ND implies \( \forall y F_f(y) \text{ diverges} \) implies \( F_f \notin K \).

Thus, ND \leq_m K
4. **AlwaysDominates(AD) = \{ f \mid \text{for all } x, f(x) > x \}**

a.) Show some minimal quantification of some known primitive recursive predicate that provides an upper bound for the complexity of AD.

\[ \forall x \exists t \ [\text{STP}(f,x,t) \land (\text{VALUE}(f,x,t) > x)] \]

b.) Use Rice’s Theorem to prove that AD is undecidable. Be Complete.

AD is non-trivial as \( S(x) = x+1 \in AD \) and \( C0(x) = 0 \notin AD \)

Let \( f,g \) be two arbitrary indices of procedures such that \( \forall x \ f(x) = g(x) \)

\( f \in AD \iff \forall x \ f(x) < x \iff \forall x \ g(x) < x \iff g \in AD \)

AD is non-trivial as \( S(x) = x+1 \in AD \) and \( C0(x) = 0 \notin AD \)

Let \( f,g \) be two arbitrary indices of procedures such that \( \forall x \ f(x) = g(x) \)

\( f \in AD \iff \forall x \ f(x) < x \iff \forall x \ g(x) < x \iff g \in AD \)

c.) Show that \( TOT = \{ f \mid \text{for all } x, f(x) \text{ converges} \} \) is many-one reducible to AD.

Let \( f \) be an arbitrary index. From \( f \), define \( \forall x \ F_f(x) = f(x) - f(x) + x + 1. \)

\( f \in TOT \) implies \( \forall x \ F_f(x) = x+1 \) implies \( F_f \in AD. \)

\( f \notin TOT \) implies \( \exists x \ F_f(x) \text{ diverges} \) implies \( F_f \notin AD. \)

Thus, \( TOT \leq_m AD \)

d.) Show that AD is many-one reducible to \( TOT = \{ f \mid \text{for all } x, f(x) \text{ converges} \} \)

Let \( f \) be an arbitrary index. From \( f \), define \( \forall x \ F_f(x) = \mu y [ f(x) > x ] \)

\( f \in AD \) implies \( \forall x \ F_f(x) \text{ converges} \) implies \( F_f \in TOT \)

\( f \notin AD \) implies \( \exists x \ F_f(x) \text{ diverges} \) implies \( F_f \notin TOT. \)

Thus, \( AD \leq_m TOT \)