Sample Question#1. Part a

1. Prove that the following are equivalent

a) S is an infinite recursive (decidable) set.

b) S is the range of a monotonically increasing total recursive function.

Note: f is monotonically increasing means that \( \forall x \ f(x+1) > f(x) \).

a) Implies b)

Let \( x \in S \iff \chi_S(x) \)
Define \( f_R(0) = \mu x \chi_S(x) \); \( f_R(y+1) = \mu x [\chi_S(x) \&\& (x > f_R(y))] \)
Clearly, since S is non-empty, it has a least one value and so there exist a smallest value such that \( \chi_S(x) \); we will enumerate this as \( f_R(0) = \mu x \chi_S(x) \).
Assume we have enumerated the y-th value in S as \( f_R(y) \). Since S is infinite, there will be values in S greater than \( f_R(y) \) and our search \( \mu x [\chi_S(x) \&\& (x > f_R(y))] \) will find the next largest value for which \( \chi_S(x) \). Thus, inductively, we will enumerate the elements of S in increasing order, as desired.
Sample Question#1 Part b

1. Prove that the following are equivalent
   a) S is an infinite recursive (decidable) set.
   b) S is the range of a monotonically increasing total recursive function.
      Note: f is monotonically increasing means that \( \forall x \ f(x+1) > f(x) \).
   b) Implies a)
   Let S be enumerated by the monotonically increasing algorithm \( f_S \).
   Define \( \chi_S \) by
   \[
   \chi_S(x) = (f_S (\mu z \ [f_S(z) \geq x]) == x)
   \]
   Clearly, if x is enumerated, it must appear before any values greater than it are enumerated and consequently this is a bounded search to find the first element listed that is at least as large as x. If this element is x, then x is in S, else it is not. The fact that \( f_S \) is monotonically increasing makes S infinite. The fact that it has a characteristic function makes it decidable.
Sample Question#2

2. Let A and B be re sets. For each of the following, either prove that the set is re, or give a counterexample that results in some known non-re set.

Let A be semi decided by \( f_A \) and B by \( f_B \)

a) \( A \cup B \): must be re as it is semi-decided by

\[
f_{A \cup B}(x) = \exists t \left[ \text{stp}(f_A, x, t) \lor \text{stp}(f_B, x, t) \right]
\]

b) \( A \cap B \): must be re as it is semi-decided by

\[
f_{A \cap B}(x) = \exists t \left[ \text{stp}(f_A, x, t) \land \text{stp}(f_B, x, t) \right]
\]

c) \( \neg A \): can be non-re. If \( \neg A \) is always re, then all re are recursive as any set that is re and whose complement is re is decidable. However, \( A = K \) is a non-rec, re set and so \( \neg A \) is not re.
Sample Question#3

3. Present a demonstration that the *even* function is primitive recursive.

   \[ \text{even}(x) = 1 \text{ if } x \text{ is even} \]
   \[ \text{even}(x) = 0 \text{ if } x \text{ is odd} \]

   You may assume only that the base functions are prf and that prf’s are closed under a finite number of applications of composition and primitive recursion.

   \[ \text{even}(0) = 1; \text{even}(y+1) = \neg\text{even}(y) = 1-\text{even}(y) \]
Sample Question#4

4. Given that the predicate \texttt{STP} and the function \texttt{VALUE} are prf’s, show that we can semi-decide

\{ f \mid \varphi_f \text{ evaluates to 0 for some input} \}

This can be shown re by the predicate

\{f \mid \exists<x,t> [ \text{stp}(f,x,t) \&\& \text{value}(f,x,t) = 0] \}
Sample Question#5

5. Let $S$ be an re (recursively enumerable), non-recursive set, and $T$ be re, non-empty, possibly recursive set. Let $E = \{ z \mid z = x + y, \text{ where } x \in S \text{ and } y \in T \}$.

(a) Can $E$ be non re? **No** as we can let $S$ and $T$ be semi-decided by $f_S$ and $f_T$, resp., $E$ is then semi-dec. by $f_E (z) = \exists <x, y, t> [\text{stp}(f_S, x, t) \&\& \text{stp}(f_T, y, t) \&\& (z = \text{value}(f_S, x, t) + \text{value}(f_T, y, t))]$

(b) Can $E$ be re non-recursive? **Yes**, just let $T = \{0\}$, then $E = S$ which is known to be re, non-rec.

(c) Can $E$ be recursive? **Yes**, let $T = \emptyset$, then $E = \{ x \mid x \geq \text{min} (S) \}$ which is a co-finite set and hence rec.
Sample Question#6

6. Assuming **TOTAL** is undecidable, use reduction to show the undecidability of

\[ \text{Incr} = \{ f \mid \forall x \phi_f(x+1) > \phi_f(x) \} \]

Let \( f \) be arb.

Define \( G_f(x) = \phi_f(x) - \phi_f(x) + x \)

\( f \in \text{TOTAL} \iff \forall x \phi_f(x) \downarrow \iff \forall x \ G_f(x) \downarrow \iff \forall x \ \phi_f(x) - \phi_f(x) + x = x \) implies \( G_f \in \text{Incr} \)

\( f \notin \text{TOTAL} \iff \exists x \phi_f(x) \uparrow \iff \exists x \ G_f(x) \uparrow \iff \exists x \ (\phi_f(x) - \phi_f(x) + x) \uparrow \) implies \( G_f \notin \text{Incr} \)
Sample Question#7

7. Let $\text{Incr} = \{ f \mid \forall x, \varphi_f(x+1) > \varphi_f(x) \}$. Let $\text{TOT} = \{ f \mid \forall x, \varphi_f(x) \downarrow \}$. Prove that $\text{Incr} \equiv_m \text{TOT}$. Note Q#6 starts this one.

Let $f$ be arb.

Define $G_f(x) = \exists t [\text{stp}(f,x,t) \& \& \text{stp}(f,x+1,t) \& \& (\text{value}(f,x+1,t) > \text{value}(f,x,t))]$

$f \in \text{Incr}$ iff $\forall x \varphi_f(x+1) > \varphi_f(x)$ iff $\forall x G_f(x) \downarrow$ iff $G_f \in \text{TOT}$
8. Let \( \text{Incr} = \{ f \mid \forall x \varphi_f(x+1) > \varphi_f(x) \} \). Use Rice’s theorem to show \( \text{Incr} \) is not recursive.

Non-Trivial as
\( C_0(x) = 0 \notin \text{Incr} \); \( S(x) = x+1 \in \text{Incr} \)

Let \( f, g \) be arb. Such that \( \forall x \varphi_f(x) = \varphi_g(x) \)
\( f \in \text{Incr} \) iff \( \forall x \varphi_f(x+1) > \varphi_f(x) \) iff
\( \forall x \varphi_g(x+1) > \varphi_g(x) \) iff \( g \in \text{Incr} \)
Sample Question#9

9. Let $S$ be a recursive (decidable set), what can we say about the complexity (recursive, re non-recursive, non-re) of $T$, where $T \subseteq S$?

Nothing. Just let $S = \emptyset$, then $T$ could be any subset of $\emptyset$. There are an uncountable number of such subsets and some are clearly in each of the categories above.
10. Define the pairing function $\langle x,y \rangle$ and its two inverses $\langle z \rangle_1$ and $\langle z \rangle_2$, where if $z = \langle x,y \rangle$, then $x = \langle z \rangle_1$ and $y = \langle z \rangle_2$.

$$\text{pair}(x,y) = \langle x,y \rangle = 2^x \ (2y + 1) - 1$$

with inverses

$$\langle z \rangle_1 = \log_2(z+1)$$

$$\langle z \rangle_2 = (((z + 1) \ // \ 2 \ \langle z \rangle_1) - 1) \ // \ 2$$
11. Assume $A \leq_m B$ and $B \leq_m C$. Prove $A \leq_m C$. In this proof, we will assume the universe for each set $S$ is $U_S$. In general $U_S = \mathbb{N}$

$A \leq_m B$ iff there exists an m-1 algorithm $f_1: U_A \rightarrow U_B$ such that $x \in A \iff f_1(x) \in B$

$B \leq_m C$ iff there exists an m-1 algorithm $f_2: U_B \rightarrow U_C$ such that $x \in B \iff f_2(x) \in C$

Define $f_3(x) = f_2(f_1(x))$, $f_3: U_A \rightarrow U_C$ is an m-1 algorithm and $x \in A \iff f_3(x) \in C$ implies $A \leq_m C$ as was desired
12. Let $P = \{ f \mid \exists x \ [ \text{STP}(f, x, x) ] \}$. Why does Rice’s theorem not tell us anything about the undecidability of $P$?

This is not an I/O property as we can have implementations of $C_0$ that are efficient and satisfy $P$ and others that do not.