Relaxed Scheduling for Scalable Belief Propagation

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ABSTRACT

The ability to leverage large-scale hardware parallelism has been one of the key enablers of the accelerated recent progress in machine learning. Consequently, there has been considerable effort invested into developing efficient parallel variants of classic machine learning algorithms. However, despite the wealth of knowledge on parallelization, some classic machine learning algorithms often prove hard to parallelize efficiently while maintaining convergence.

In this paper, we focus on efficient parallel algorithms for the key machine learning task of inference on graphical models, in particular on the fundamental belief propagation algorithm. We address the challenge of efficiently parallelizing this classic paradigm by showing how to leverage scalable relaxed schedulers in this context. We present an extensive empirical study, showing that our approach outperforms previous parallel belief propagation implementations both in terms of scalability and in terms of wall-clock convergence time, on a range of practical applications.

1 INTRODUCTION

Hardware parallelism has been the key computational enabler of the recent advances in machine learning, as it provides a way to reduce the processing time for the ever-increasing quantities of data required for training accurate models. Therefore, naturally, there has been a considerable amount of effort invested into developing efficient parallel variants of classic machine learning algorithms, e.g. [17, 25–27, 31].

In this paper, we will focus on efficient parallel algorithms for a fundamental machine learning task of *inference on graphical models*. Specifically, graphical models [23] are a general framework for describing the statistical relationships between large collections of random variables. The inference task in graphical models takes the form of *marginalisation*: we are given observations for a subset of the random variables, and the task is to compute the conditional distribution of one or a few variables of interest. The marginalization problem is known to be computationally intractable in general; it is NP-hard and difficult to even approximate [10, 11, 36], which has lead to use of inexact heuristics in practical inference tasks.

One popular heuristic for inference on graphical models is *belief propagation* [30], inspired by the exact dynamic programming algorithm for marginalization on trees. While belief propagation has no general approximation or even convergence guarantees, it has proven empirically successful in inference tasks, in particular in the context of decoding low-density parity check codes [9]. However, it remains poorly understood how to properly parallelize belief propagation.

Parallelizing Belief Propagation. To illustrate the challenges of parallelizing belief propagation, we will next give a simplified overview of the belief propagation algorithm, and refer the reader to Section 2 for full details. Belief propagation can be seen as a *message passing* or a *weight update* algorithm. In brief, belief propagation operates over the underlying graph $G = (V, E)$ of the graphical model, maintaining a vector of real numbers called a *message* $\mu_{i\to j}$ for each ordered pair (i, j) corresponding to an edge $\{i, j\} \in E$ (Fig. 1). The core of the algorithm is the *message update rule* which specifies how to update an outgoing message $\mu_{i\to j}$ at node i based on the *other* incoming messages at node i; for the purposes of the present discussion, it is sufficient to view this as black box function f over these other messages, leading to the update rule

$$\mu_{i\to j} \leftarrow f\big(\{\mu_{k\to i} : k \in N(i) \setminus \{j\}\}\big)\,. \tag{1}$$

This update rule is applied to messages until the values of messages have converged to a stable solution, at which point the algorithm is said to have terminated.

Importantly, the message update rule does not specify *in which order* messages should be updated. The standard solution, called *synchronous belief propagation*, is to update all the message simultaneously. That is, in each global round $t = 1, 2, 3, \ldots$, given message values $\mu^t_{i\to j}$ for all pairs (i, j), the new values $\mu^{t+1}_{i\to j}$ are computed as

$$\mu^{t+1}_{i\to j} \leftarrow f\big(\{\mu^t_{k\to i} : k \in N(i) \setminus \{j\}\}\big)$$

However, there is strong evidence suggesting that updating messages *one at a time* leads to faster and more reliable convergence [15]; in particular, various proposed *priority-based schedules*—schedules that try to prioritize message updates that would make 'more progress'—have proven to converge with much fewer message updates than the synchronous schedule in empirical trials [15, 22, 41].

Having to execute updates in a strict priority order poses a challenge for efficient *parallel* implementations of belief propagation: while the synchronous schedule is naturally parallelizable, as all message updates can be done independently, the more efficient priority-based schedules are inherently sequential and thus seem difficult to parallelize. Accordingly, existing work on efficient parallel belief propagation has focused on designing custom schedules that try to import some features from the priority-based schedules while maintaining a degree of parallelism [12, 17].

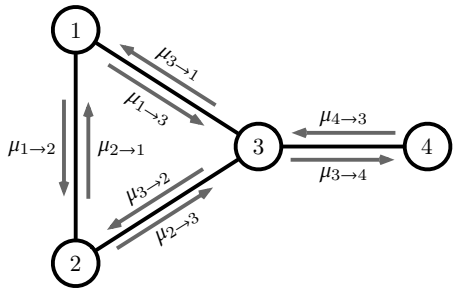


Figure 1: State of the belief propagation algorithm consist of two directed messages for each edge.

1.1 Our contributions

In this work, we address the challenges of parallel belief propagation by showing how to efficiently parallelize any priority-based schedule for belief propagation. The key idea is that we can *relax* the priority-based schedules by allowing limited out-of-order execution, concretely implemented using a *relaxed scheduler*, as we will explain next.

More precisely, consider a belief propagation algorithm that schedules the message updates according to a priority function r by always updating the message $\mu_{i\to j}$ with the highest priority $r(\mu_{i\to j})$ next; this framework captures existing priority-based schedules such as residual belief propagation [15] and its variants [22, 41]. Concretely, a iterative centralized version of this algorithm can be implemented by storing the messages in a priority queue Q, and iterating the following procedure:

(1) Pop the top element for Q to obtain the message $\mu_{i\to j}$ with highest priority $r(\mu_{i\to j})$.
(2) Update message $\mu_{i\to j}$ following (1).
(3) Update the priorities in Q for messages affected by the update.

As one can readily see, this template does not directly lend itself to efficient parallel implementation. Previous work, e.g. [12, 17] investigated various heuristics for the parallel scheduling of updates in belief propagation, trading off increased parallelism with additional work in processing messages or even potential loss of convergence.

In this paper, we investigate an alternative approach, replacing the priority queue Q with a *relaxed scheduler* to obtain a efficient parallel version of the above template. Specifically, the relaxed scheduler is a data structure with similar semantics to those of a priority queue, but instead of guaranteeing that the top element is always returned first in the linearized order, it only guarantees to return one of the top k elements of the priority queue, where k is a variable parameter. Relaxed schedulers have recently become popular in the context of parallel graph processing frameworks, e.g. [18, 29], where it has been shown that they can induce nontrivial trade-offs between the degree of relaxation and the scalability of the underlying implementation, e.g. [1, 5]. In the context of belief propagation, this method induces a *relaxed priority-based scheduling* of the messages, roughly following the original schedule but allowing for message updates to be performed out of order. In this paper, we investigate the convergence-scalability trade-off when applying relaxed scheduling to residual belief propagation, from both theoretical and practical standpoints.

Experimental evaluation. We implement our relaxed priority-based scheduling framework with a *Multiqueue* data structure [34] and instantiate it with several proposed priority-based schedules, including residual belief propagation [15], the priority-based schedule of [41], the weight decay algorithm of [22] and the residual splash algorithm [17]; the latter requires a slight extension of our framework, as we discuss below.

In the benchmarks, we show that this framework gives state-of-the-art parallel scalability on a wide variety of Markov random field models. As can be expected, the relaxed priority-based schedules require slightly more message updates than their exact counterparts, but this is offset by the better scalability of relaxed schedulers. Indeed, we highlight the fact that the relaxed version of the popular residual belief propagation algorithm performs extremely well in both single-thread and highly parallel regimes, making it an attractive practical solution for belief propagation.

2 PRELIMINARIES AND RELATED WORK

2.1 Inference on graphical models

We consider marginalization in *pairwise Markov random fields*; note that one can equivalently consider factor graphs or Bayesian networks [43]. A pairwise Markov random field is defined by a set of random variables $X_1, X_2, \ldots, X_n$, a graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$, and a set of *factors*

$$\psi_i \colon D_i \to \mathbb{R}^+ \qquad \text{for } i \in V,$$
$$\psi_{ij} \colon D_i \times D_j \to \mathbb{R}^+ \qquad \text{for } \{i,j\} \in E,$$

where D_i denotes the domain of random variable X_i. The edge factors ψ_{ij} represent the dependencies between the random variables, and the node factors ψ_i represent a priori information about the individual random variables; the Markov random field defines a joint probability distribution on $X = (X_1, X_2, \ldots, X_n)$ as

$$\Pr\big[X = x\big] \propto \prod_i \psi_i(x_i) \prod_{ij} \psi_{ij}(x_i, x_j)\,,$$

where the 'proportional to' notation $\propto$ hides the normalization constant applied to the right-hand side to obtain a probability distribution. Formally, the marginalization problem is to compute the probabilities $\Pr[X_i = x]$ for a specified subset of variables; for notational convenience, we assume that any possible observations regarding the values of other random variables are encoded in the node factor functions ψ_i. The marginalization problem is known to be computationally intractable; it is NP-hard and difficult to even approximate [10, 11, 36], which has lead to use of inexact heuristics, such as belief propagation, in practical inference tasks.

2.2 Belief propagation

Belief propagation is a message-passing algorithm; for each ordered pair (i, j) such that $\{i, j\} \in E$, we maintain a *message* $\mu_{i\to j} \colon D_j \to \mathbb{R}$, and the algorithm iteratively updates these messages until the values (approximately) converge to a fixed point. On Markov random fields, the message update rule gives the new value of message $\mu_{i\to j}$ as a function of the old messages directed to node i by

$$\mu_{i\to j}(x_j) \propto \sum_{x_i \in D_i} \psi_i(x_i)\psi_{ij}(x_i, x_j) \prod_{k \in N(i)\setminus\{j\}} \mu_{k\to i}(x_i)\,, \quad (2)$$

where $N(j)$ denotes the neighbors of node j in the graph G. Once the algorithm has converged, the marginals are estimated as

$$\Pr[X_i = x_i] \propto \psi_i(x_i) \prod_{j \in N(i)} \mu_{j\to i}(x_i)\,.$$

Again, we note that belief propagation can be equivalently formulated on factor graphs or Bayesian networks [43].

As already discussed in the introduction, the update rule (2) can be applied in arbitrary order. The standard *synchronous belief propagation* updates all the message simultaneously; in each global round $t = 1, 2, 3, \ldots$, given message values $\mu^t_{i\to j}$ for all pairs (i, j),

the new values $\mu_{i\to j}^{t+1}$ are computed as

$$\mu_{i\to j}^{t+1}(x_j) \propto \sum_{x_i \in D_i} \psi_i(x_i)\psi_{ij}(x_i, x_j) \prod_{k \in N(i)\setminus\{j\}} \mu_{k\to i}^{t}(x_i)\,.$$

2.3 Asynchronous belief propagation

Starting with Elidan et al. [15], there has been a line of research arguing that *asynchronous* or *iterative* schedules for belief propagation tend to converge more reliably and with fewer message updates that the synchronous schedule. In particular, the practical work has focused on developing schedules that attempt to iteratively perform 'the most useful' update at each step; the most prominent of these algorithms is the *residual belief propagation* of Elidan et al. [15], with other proposals aiming to address the shortcomings of residual belief propagation in various cases.

Residual belief propagation. Given a current state of messages, let $\mu'_{i\to j}$ denote the message we would obtain by applying the message update rule (2) to message $\mu_{i\to j}$. In residual belief propagation, the priority of a message is given by the *residual* $\mathrm{res}(\mu_{i\to j})$ of a message $\mu_{i\to j}$, defined as

$$\mathrm{res}(\mu_{i\to j}) = \|\mu'_{i\to j} - \mu_{i\to j}\|\,, \tag{3}$$

where $\|\cdot\|$ is an arbitrary norm; in this work, we assume L^2 norm is used unless otherwise specified. That is, the residual of a message corresponds to amount of change that would happen if message $\mu_{i\to j}$ would be updated. Note that this means that residual belief propagation performs *lookahead*, that is, the algorithm precomputes the future updates before applying them to the state of the algorithm.

Weight decay belief propagation. Weight decay belief propagation of [22] is a variant of residual belief propagation that penalizes message priorities for repeated updates. That is, let $m(\mu_{i\to j})$ denote how many times message $\mu_{i\to j}$ has been updated by the algorithm, and let $\mathrm{res}(\mu_{i\to j})$ denote the residual of a message as above. The priority function of weight decay belief propagation is

$$r(\mu_{i\to j}) = \frac{\mathrm{res}(\mu_{i\to j})}{m(\mu_{i\to j})}\,.$$

The motivation behind this weight decay scheme is that empirical observations suggest that one possible failure mode of residual belief propagation is getting stuck in cycles with large residuals; the weight decay prioritizes other edges in cases where this happens.

Residual without lookahead. Another variant of residual belief propagation is the lookahead-avoiding belief propagation of [41]. As the name implies, this algorithm does not perform the exact residual computation using (3), but instead approximates the residuals indirectly, with the aim of reducing the computational cost of priority updates.

Informally, the basic idea is that for each message $\mu_{i\to j}$, we track the amount other incoming messages at node i have changed since the last update of $\mu_{i\to j}$, and use this to define the priority of updating $\mu_{i\to j}$. The actual approximation in the algorithm uses a slightly different notion of residual from (3), so we refer to [41] for full details.

2.4 Parallel belief propagation

As discussed above, the question of parallelizing belief propagation is fairly poorly understood. The synchronous schedule is trivially parallelizable by performing updates within each round in parallel, but the improved converge properties of the iterative schedules cannot easily be translated to parallel setting. There have been recent proposals that aim to bridge this gap in an ad-hoc manner by designing custom algorithms for specific parallel computation settings.

Residual splash. The *residual splash* belief propagation [17] is a vertex-based algorithm inspired by residual belief propagation. The residual splash algorithm was initially designed for MapReduce computation, and it aims to have larger individual tasks while retaining a similar structure to residual belief propagation.

Specifically, the residual splash algorithm works by defining a priority function over nodes of the Markov random field, and selecting the next node to process in a strict priority order. For the selected node, the algorithm performs a *splash* operation that propagates information within distance H in the graph; in practice, this results in threads performing larger individual tasks at once, offsetting the cost of accessing the strict scheduler.

In detail, the priority of for nodes is given by the *node residual*, defined as

$$\mathrm{res}(i) = \max_{j \in N(i)} \mathrm{res}(\mu_{j\to i})\,.$$

Given a *depth parameter* H, the splash operation at node i is defined by following sequence of message updates:

(1) Construct a BFS tree T of depth H rooted at node i.
(2) In the reverse BFS order on T—starting from leaves—process all nodes in T, updating all outgoing messages for each node processed.
(3) Repeat the previous step in BFS order, i.e., starting from the root.

In other words, this process gather all available information at radius H from the selected node, and propagates it to all nodes within the radius.

Randomized synchronous belief propagation. Van der Merve et al. [12] proposed a parallelization scheme for belief propagation on GPUs, mixing the structure of synchronous and residual belief propagation. Their algorithm considers all messages at once in global rounds, and performs the following filter-and-select steps before computing the message updates:

(1) Filter out all messages whose residuals are below the convergence threshold.
(2) Out of the remaining messages, select a p fraction of messages uniformly at random to update.

Alternatively, the process can perform the algorithm on per-node basis, using node residuals as in the residual splash algorithm.

The fraction p is adjusted on the fly based on the convergence of the algorithm, preferring a low value if the algorithm is converging slowly, and a high value if it is converging fast. Concretely, the selection scheme for p used by [12] is to set $p = 1$ if the number of messages above the convergence threshold decreased by at least 10% in the last round, and set it to a smaller fixed value otherwise.

We note that the randomized synchronous algorithm is particularly well suited for GPU use, as the filter-and select steps can be efficiently implemented on GPUs. However, as shown by our experimental study, this strategy is not efficient on a subset of real-world models, when ported to CPU. Conversely, as discussed by the authors of [12], the dynamic priority-based strategy we propose would be hard to implement efficiently on GPUs, due to its irregular structure.

2.5 Relaxed schedulers

Recently, significant amount of attention has been given to parallelizing iterative algorithms, especially in the context of large-scale graph processing, e.g. [8, 13, 14, 18, 29]. One way to exploit the fine-grained parallelism present in these applications has been to analyze and leverage their shallow dependency structure, e.g. [7, 40]. While this approach can provide theoretical guarantees and may yield extremely strong practical results, it does require a good understanding of the problem at hand, as well as potential tuning of parameters in practice.

An alternative approach has been to employ scalable data structures which only ensure *relaxed priority order* to schedule the iterations. To our knowledge, this idea was first proposed by Karp and Zhang [21] in the context of parallel backtracking in the PRAM model, who noticed that the scheduler may relax the strict order induced by the sequential algorithm, allowing tasks to be processed speculatively ahead of their dependencies, without loss of correctness. A simple instance of this phenomenon is in the context of single-source shortest-paths (SSSP), where the scheduler may retrieve vertices in arbitrary order without breaking correctness, as the distance at each vertex is guaranteed to eventually converge to the minimum, although executing items too far from the sequential order may be wasteful in terms of tasks processed, and thus negate the benefits of parallelism. See e.g. [28] for a more in-depth treatment in the case of SSSP.

More generally, this relaxed approach has been quite popular in practice, as several efficient relaxed schedulers as well as applications have been proposed [3, 4, 6, 19, 29, 35, 38, 39, 42]. Such frameworks can attain state-of-the-art results in the graph processing domain [18, 20, 29]. A parallel line of work has attempted to provide guarantees on the amount of relaxation in individual schedulers [2, 3, 37], as well as the impact of using relaxed scheduling on existing iterative algorithms [1, 5]. In this paper, we are employing the modeling of relaxed schedulers used in e.g. [2, 5] for graph algorithms, but applying it to a new domain, inference on graphical models. We will see in Section 4 that obtaining general bounds on the impact of relaxation on the work performed by the algorithm, along the lines of those obtained by [5] for graph algorithms, is infeasible for belief propagation.

3 RELAXED PRIORITY-BASED BELIEF PROPAGATION

In this section, we describe our framework for parallelizing belief propagation schedules via relaxed schedulers. The main idea of the framework follows the description given in Section 1.1; however, we generalize it slightly to capture schedules that do not use individual messages as elementary tasks, such as the residual splash algorithm [17].

3.1 Priority-based belief propagation

Given a Markov random field, a priority-based schedule for belief propagation is defined by a set of *task* $T_1, T_2, \ldots, T_K$, each corresponding to a sequence of edge updates, and a priority function r that assigns a priority $r(T_i)$ to a task based on the current state of the messages as well as possible auxiliary information maintained separately. A priority-based schedule is executed sequentially by repeating the following steps until a convergence criterion is reached:

(1) Select the task T_i with highest priority $r(T_i)$
(2) Perform all message updates specified by the task T_i.
(3) Update the priorities for all tasks.

Note that tasks can be executed multiple times in this framework. In particular, we assume that the priority $r(T_i)$ of a task T_i can only remain the same or increase when other tasks are executed, and the only point where the priority decreases is when the task is actually executed.

3.2 Implementation

Concretely, a sequential version of a priority-based schedule for belief propagation can be implemented using a priority queue Q. The queue Q stores entries for all individual tasks, and the top priority task is obtained by popping the top element of the queue. The changes to other task priorities are updated using the increase key operation of the queue.

One could map this sequential pattern directly to a parallel setting by replacing the sequential priority queue with a linearizable concurrent one. However, this may not be the best option, for two reasons. First, in general it is challenging to build *scalable* linearizable exact priority queues, see e.g. [24]—the data structure is inherently contended, which leads to poor cache behavior. Second, in this context, linearizability only gives the illusion of atomicity with respect to task message updates: the data structure only ensures that the *task removal* is atomic, whereas the actual message updates which are part of the task are not usually performed atomically together with the removal.

For these reasons, in our framework, we opt for using a variant of the Multiqueue relaxed priority scheduler [3, 34]. More precisely, we assume that each thread i has one or a few local concurrent priority queues, used to store pointers to tasks, prioritized by an algorithm-specific function. (In our experiments, we use binary heaps, protected by coarse-grained locks, for these priority queues.) Additionally, we store additional metadata as required by the algorithm and the graphical model. To process a new task, the thread selects two among all the priority queues uniformly at random, and withdraws the task from the queue whose top element has higher priority. (The task is marked as in-process so it cannot be processed concurrently by some other thread.) The thread then proceeds to perform the metadata updates required by the underlying variant of belief propagation. The termination condition is checked periodically once the number of iterations reaches a predefined threshold.

Concretely, for *relaxed residual belief propagation*, which is or main algorithmic proposal, the tasks are messages, prioritized by their residual values. Initially, we insert one task per message, assigning them into the priority queues via a fixed hash function, so that messages are easily addressable. We maintain a lock for each node as metadata, and a task for each message. Tasks never get removed from their priority queues, although they can be deprioritized. Whenever processing a task corresponding to message $\mu_{i\to j}$, we first lock nodes i and j (in a fixed order, to avoid deadlocks), update the message, and then update all the priorities of messages from node j accordingly. Finally, we reset the priority of message $\mu_{i\to j}$ to zero, and unlock the message endpoints. The termination condition in this case is if the residual of the highest-priority message is below a fixed tolerance level, e.g. 10^{-5}.

4 DYNAMICS OF RELAXED BELIEF PROPAGATION

As we see in Section 5, the relaxed versions of priority-based belief propagation schedules yield fast converge times on a wide variety of Markov random fields; specifically, the number of message updates is roughly the same as for the non-relaxed version, while the running times are lower. The complementary theoretical question is whether we can give analytical bounds how much extra work—in terms of additional message updates—the relaxation incurs.

Unfortunately, the dynamics of even synchronous belief propagation are poorly understood on general graphs, and none of the priority-based algorithms provide general guarantees on the convergence time. As such, we can only hope to gain some limited understanding on why relaxation retains the fast convergence properties of the exact priority-based schedules. Here, we present a limited theoretical evidence suggesting that as long as a schedule does not impose long dependency chains in the sequence of updates, then relaxation incurs relatively little overhead.

4.1 Sequential model for relaxed schedules

For analysis of the relaxed priority-based belief propagation, we consider the formal sequential model introduced by [2, 5] to analyze performance of iterative algorithms under relaxed schedulers. Specifically, we model a relaxed scheduler Q_k as a data structure which stores pairs corresponding to tasks and their priorities, and supports the following operations:

- *Pop* returns one of the k highest priority elements in Q_k and removes it from Q_k.
- *Insert* inserts an task with a specified priority into Q_k.
- *IncreaseKey* increases the priority of a specified task in Q_k.

Given a sequence of operations for Q_k, we assume that resulting execution satisfies the following axioms [2, 5]:

- Each *Pop* operation returns one of the k highest priority elements in Q_k.
- Assume that a task T becomes the highest priority task in Q_k at some point during the execution. Unless there is an *Insert* or *IncreaseKey* operation that replaces T as the highest priority element, then one of the next k *Pop* operations returns T.

Other than satisfying these axioms, we assume that the behavior of Q_k can be adversarial, or randomized. In particular, it is known that executions of the Multiqueue data structure [34] satisfy these axioms with high probability [1, 3].

We model the behavior of relaxed priority-based belief propagation by investigating the number of message updates needed for convergence when the algorithm is executed *sequentially* using a relaxed scheduler Q_k satisfying the above constraints. We emphasize that this analysis reduces to a sequential game between the algorithm, which queries Q_k for tasks/messages, and the scheduler, which returns messages in possibly arbitrary fashion. One may think of the relaxed sequential execution as a form of linearization for the actual parallel execution—reference [1] formalizes this intuition.

4.2 Relaxed schedules on trees

We now consider the behavior of relaxed residual belief propagation schedules on *trees with a single source*, similarly to the analysis of residual splash given by Gonzalez et al. [17]. Specifically, assume that the Markov random field and the initialization of the algorithm satisfies the following conditions:

- The graph $G = (V, E)$ is a tree with a specified root r.
- The factors of the Markov random field and the initial messages are such that the residuals are zero for all messages other than the outgoing messages from the root, i.e., $\text{res}(\mu_{i\to j}) = 0$ if $i \neq r$.

These conditions mean that residual belief propagation will start from the root, and propagate the messages down the trees until propagation reaches all leaves. In particular, residual belief propagation without relaxation will perform $n - 1$ message updates before convergence, updating each message away from root once. While this is a restrictive setting, we note that there are practical inference instances where the Markov random field has locally tree-like structure, e.g. LDPC codes (see Section 5).

To characterize the dynamics on relaxed residual belief propagation on trees with a single source, we observe that the algorithm can make two types of message updates:

- Updating a message with zero residual, in which case nothing happens (*a wasted update*). This happens if the scheduler relaxes past the range of messages with non-zero residual.
- Updating a message $\mu_{i\to j}$ with non-zero residual, in which case the residual of $\mu_{i\to j}$ goes down to zero, and the messages $\mu_{j\to k}$ for the children k of j may change their residuals to non-zero values (*a useful update*).

It follows that each edge will get updated only once with non-zero residual. At any point of time during the execution of the algorithm, we say that the *frontier* is the set of messages with non-zero residual, and use $F(t)$ to denote the size of the frontier at time step t.

To see how the size of the frontier relates to the number message updates in relaxed residual belief propagation, observe that after a useful update, we have one of the following cases:

- If $F(t) \geq k$, then the next *Pop* operation to Q_k will give an edge with non-zero residual, resulting in a useful update.
- If $F(t) < k$, then in the worst case we need k *Pop* operations until we perform a useful update.

We next discuss how to use this to analyse the extra work incurred by relaxation in two concrete cases.

Good case: uniform expansion. As the first case, we consider the tree model in the case where the edge factors ψ_{ij} are identical for all edges and not *deterministic*, i.e. $\psi_{ij}(x_i, x_j) \neq 0$ for all $\{i, j\}$. Let us say that the *level* of a message $\mu_{i\to j}$ is ℓ if the distance from i to the root r is ℓ. The conditions we imposed our Markov random field, together with the update rule (2), imply that the residuals of the messages are decreasing in the level ℓ of the message, and all messages on level ℓ will have the same residual when they are in the frontier. This means that residual schedule will prefer updating messages on lower levels first.

Now consider the progress of the relaxed residual belief propagation on this tree; let H denote the height of the tree. Now assume that all messages on levels $0, 1, \ldots, \ell - 1$ have had a useful update, and consider how many wasted updates we can make in the worst case before all messages on level ℓ have been processed. Let f denote the number of message on level ℓ still in the frontier:

- While $f \geq k$, there are at least k messages of level ℓ on the frontier. Since they have the highest residual out of the messages in the frontier, each update is a useful update of a message on level ℓ.
- When $f < k$, there can be updates that do not hit messages on level ℓ, which can possibly be wasted updates. However, the highest-priority messages are still from level ℓ, so every kth update will hit a message on level ℓ by the guarantees of the scheduler. Thus, in $(k-1)f = O(k^2)$ updates, all remaining messages on level ℓ have been processed.

Since there can be at most $n - 1$ useful updates, and the number of levels is $H - 1$, the total number of updates performed by relaxed residual belief propagation is $n + O(Hk^2)$.

Bad case: long paths. A simple example where relaxed residual belief propagation performs poorly is a path. That is, if our underlying tree is a path of length n with a root at one end, then relaxed residual belief propagation can perform $\Omega(kn)$ message updates in the worst case. However, the path has height $H = n$, so one might ask if there is a general upper bound of form $n + O(Hk^2)$ on trees without restricting the edge factors as in our previous example.

Unfortunately, turns out that without the restrictions above, we can construct examples of trees with height $H = o(n)$ where relaxed residual belief propagation still performs $\Omega(kn)$ message updates (see Figure 2 for an illustration):

(1) Start with a path of length $\sqrt{n}$, with a root at one end.
(2) Attach a new path of length $\sqrt{n}$ to each vertex.
(3) For each remaining degree-2 node in the graph, attach a single new node to it.

This construction results in a 3-regular rooted tree with $\Theta(n)$ nodes and depth $H = O(\sqrt{n})$. Finally, we choose the edge factors so that residuals on the side paths are larger than the residuals on the main path, so residual belief propagation will prefer following the side paths first.

One can now observe that under the adversarial model for the relaxed scheduler, the adversary can select the execution of the relaxed scheduler so that the frontier size never exceeds 4. That is, adversary forces the algorithm to process the graph one side path at time, wasting $k - 1$ steps between each useful update.

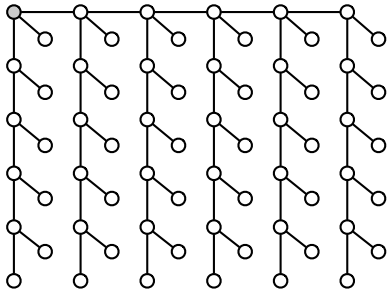


Figure 2: Example of the tree construction where relaxed residual belief propagation performs poorly.

Finally, we note that the same construction can be generalized to obtain instances with similar relaxation overhead and diameter $O(n^{1/c})$ for larger constants $c < k$, by simply working with paths of length $n^{1/c}$ and repeating the path attachment step c times.

Remark 1. As suggested by the above examples, one might consider changing the priority function to preferentially select messages closer to the source. This can lead to improved work guarantees for the relaxed schedule. Indeed, we discuss one concrete example in Appendix A, where we show how to relax the optimal schedule on trees. However, it is not straightforward to construct such priority functions so that they also make sense on general graphs, which can have non-monotonic potentials and cycles.

5 EVALUATION

5.1 Algorithms

For the experiments, we have implemented multiple priority-based algorithms and instantiated them with both exact and relaxed priority schedulers.

Priority-based algorithms. We implemented the residual belief propagation, weight decay belief propagation and residual without lookahead as described in Section 2. For the residual splash, we implemented two variants. The first is the standard splash algorithm, and the second is an optimized version we refer to as *smart splash*, which only updates messages along the BFS edges during a splash operation; the latter variant has similar convergence times as the baseline residual splash algorithm, but performs fewer message updates.

We include the following instantiations of the algorithms in the benchmarks:

- *Exact scheduling:* We include the exact versions of residual belief propagation (Coarse-Grained Residual) and both residual splash (Splash) and smart splash (Smart Splash) with $H \in \{2, 5, 10\}$. For these algorithms, the scheduler is a standard concurrent priority queue. The other exact priority-based algorithms are not included, as they generally perform worse on our test instances.
- *Relaxed scheduling:* We include the relaxed versions of residual belief propagation (Relaxed Residual), weight decay belief propagation (Weight-decay), residual without lookahead

(Relaxed Priority) and smart splash (Relaxed Smart Splash) with $H \in \{2, 5, 10\}$. For these algorithms, the scheduler is a Multiqueue with 4 priority queues per thread, as discussed in Section 3.

Baselines. As a baseline, we implemented a parallel version of the standard synchronous belief propagation (Synchronous) and the randomized synchronous belief propagation of Van der Merve et al. [12]. The latter is omitted from the benchmarks, as our CPU-based implementation of this GPU-based algorithm did not perform well, even after significant tuning. (See Appendix B for detailed discussion.)

5.2 Models

We run our experiments on four Markov random fields models.

Trees. As a simple base case, we consider a simple tree model similar to the analytical setting in Section 4. The underlying graph is a full binary tree on 1 million vertices, and the other parameters are set up as follows:

- All variables are binary, i.e. the domain is $\{0, 1\}$ for each variable.
- Vertex factors are $(0.1, 0.9)$ for the root and $(0.5, 0.5)$ for all other vertices.
- Edge factors are $\psi_{ij}(x, y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$ for all edges.

As discussed in Section 4, these choices create a setup where the belief propagation has to propagate information from the root to all other nodes. Thus, under an optimal schedule, the total number of performed updates is be equal to $10^6 - 1$. Since we know that all algorithms will converge on this model, we run the algorithms until exact convergence.

Ising and Potts models. Ising and Potts models are Markov random fields defined over an $n \times n$ grid graph, arising from applications in statistical physics. Both of Ising [15, 22] and Potts [41] models were used in prior work as test case, and in general they offer a class of good test instances, as they both exhibit complex cyclic propagations and are easy to generate.

For the parameters of the models, we mostly follow prior work in the setup. As the underlying graph, we use a 300×300 grid graph to get instances where the effects of parallelization are clearly visible. For the Ising model, we select the factors similarly to [15, 22]:

- The variable domain is $\{-1, 1\}$ for all variables.
- Vertex factors are $\psi_i(x) = e^{\beta_i x}$.
- Edge factors are $\psi_{ij}(x, y) = e^{\alpha_{ij} xy}$.
- The parameters α_{ij} and β_i are chosen uniformly at random from $[-1, 1]$.

For the Potts model, we select the factors following [41]:

- The variable domain is $\{0, 1\}$ for all variables.
- Vertex factors are $\psi_i(x) = \begin{cases} e^{\beta_i}, & x = 1 \\ 1, & x = 0 \end{cases}$.
- Edge factors are $\psi_{ij}(x, y) = \begin{cases} e^{\alpha_{ij}}, & x = y \\ 1, & x \neq y \end{cases}$.
- The parameters α_{ij} and β_i are chosen uniformly at random from $[-2.5, 2.5]$.

For both Ising and Potts models, we set the convergence threshold to 10^{-5}. That is, we terminate algorithm once all task have priority below this threshold.

LDPC codes. Finally, we generate Markov random fields corresponding to the (3, 6)-LDPC (*low density parity check code* [16]) decoding. LDPC decoding is one of the more successful application of belief propagation. We consider a simple version of LDPC decoding task where convergence guarantees exist [32]. However, we stress that coding theory is its own extensive research area, and far more optimized codes and decoding algorithms exist in practice—we simply use LDPC decoding to observe the comparative scaling behavior of our implementations on instances where synchronous belief propagation is guaranteed to converge. For a more detailed background on LDPC decoding and other aspects of coding theory, refer e.g. to the book [33].

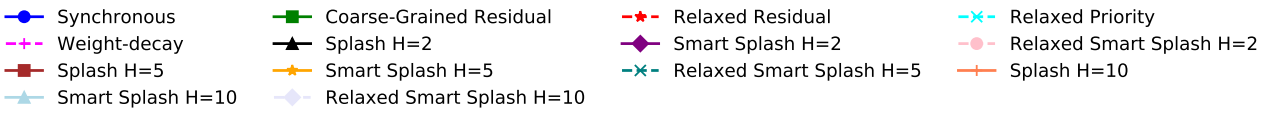
More precisely, we consider (3, 6)-LDPC decoding over a binary symmetric channels. Informally, a (3, 6)-LDPC code is a (3, 6)-regular bipartite graph, where each degree 3 node corresponds to a binary *variable* and each degree 6 node corresponds to a *constraint* of form $x_{i_1} + x_{i_2} + \ldots + x_{i_6} = 0$ over the neighboring variables $x_{i_1}, x_{i_2}, \ldots, x_{i_6}$. Each sequence of variables that satisfies the all the constraints is *codeword* of the code. The basic setup is then that we send a codeword over a *channel* that flips each bit with probability ε, and the receiver will run belief propagation and use results of marginalization to infer the original codeword.

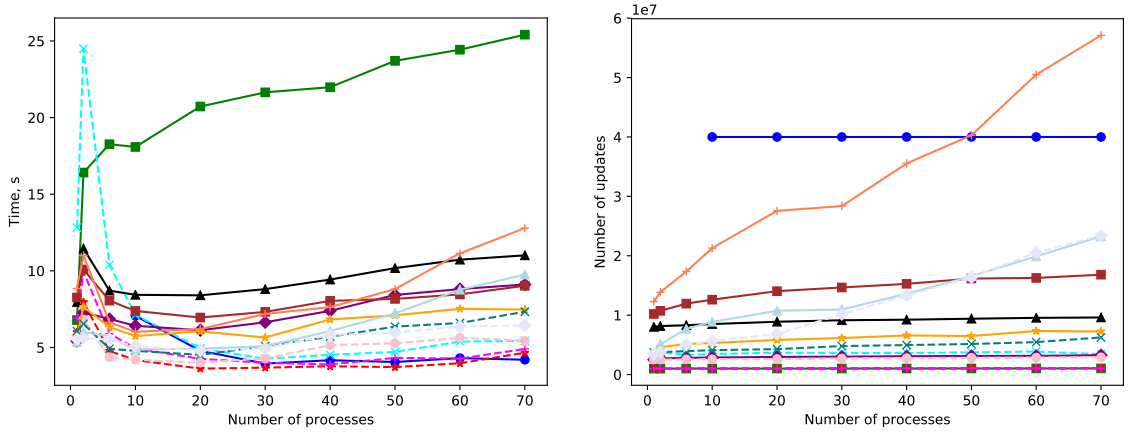
For our experiments, we build a (3, 6)-LDPC instance with 30 000 variable nodes and 15 000 constraint nodes by selecting a random (3, 6)-regular bipartite graph, and initialize the node factors corresponding to the all-zero codeword sent over binary symmetric channel with error probability $\varepsilon = 0.07$. Under these conditions, belief propagation is guaranteed to correctly decode the instance with high probability [32]; indeed, all the algorithms that converged decoded the codeword correctly in our experiments. The codeword length was again selected to get roughly comparable baseline running times as for the other instances.

Concretely, we get Markov random field where the underlying graph is a random bipartite graph with 45 000 nodes. For each variable node i, let $x_i \in \{0, 1\}$ be the 'transmitted' value of the variable, randomly generated to be 1 with probability ε and 0 otherwise. The factors have the following structure:

- The domains of variable nodes are binary domains $\{0, 1\}$. For the constraint nodes, the domain is $\{0, 1\}^6$—different bit masks of length 6.
- The node factors for variable nodes are
$$\psi_i(y) = \begin{cases} 1 - \epsilon, & y = x_i \\ \epsilon, & y \neq x_i. \end{cases}$$
For the constraint nodes, the node factor $\psi_c(y)$ is equal to the number of ones in $y \in \{0, 1\}^6$ modulo 2; this effectively penalizes any value that does not satisfy the constraint.
- Edge factors $\psi_{ic}(x, y)$ is one if the corresponding bit in the $y \in \{0, 1\}^6$ equals $x \in \{0, 1\}$, and is zero otherwise.

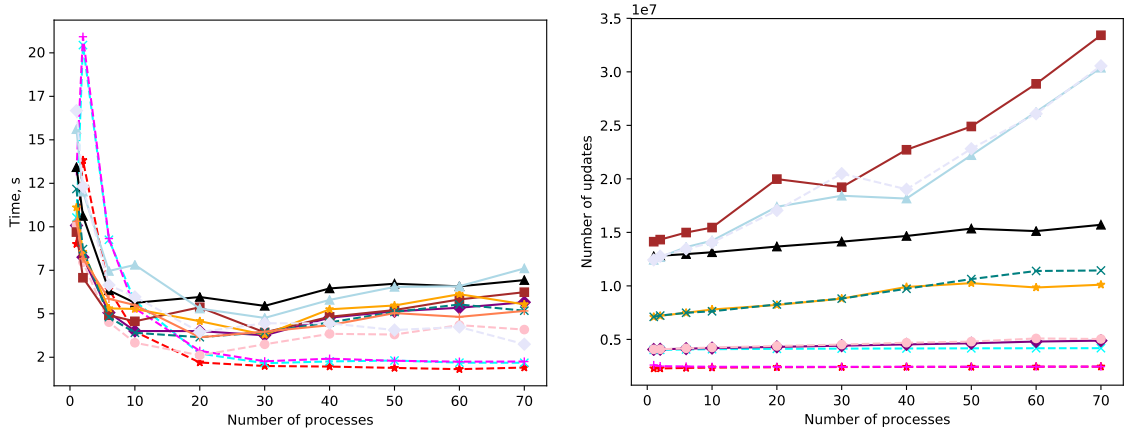
For the LDPC instances, we set the convergence threshold to 10^{-2} to ensure fast convergence; this approximates the behavior of actual LDPC decoders.





(a) Execution time **(b) Number of updates**

Figure 3: The results of the evaluation of the algorithms on the Tree model



(a) Execution time **(b) Number of updates**

Figure 4: The results of the evaluation of the algorithms on Ising model

5.3 Implementation

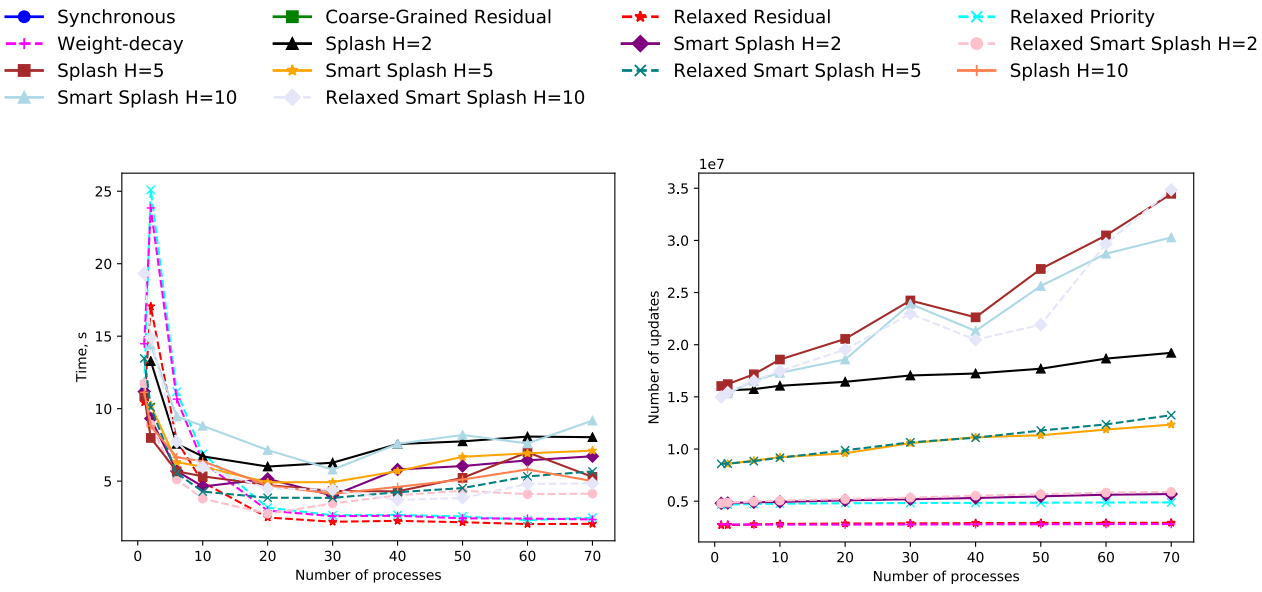
For each pair of algorithm and model, we run the experiment five times, and average the execution time and the number of performed updates on the messages. We execute on a 4-socket Intel Xeon Gold 6150 2.7 GHz server with 18 threads per socket and 512GB of RAM. The code is written in Java; we use Java 11.0.5 and OpenJDK VM 11.0.5. Our code is available at https://cutt.ly/spaa202058.

5.4 Results: scaling

How to read the plots. There are two types of plots per each model: the first shows the execution time of the algorithms, while the other one shows the number of updates performed. On the x axis we have the number of threads the algorithms were run on, while on the y axis we have: the time in seconds (for time plots) and the number of updates (for update plots). The dashed lines on the plots correspond to the algorithms that use a relaxed scheduler, while the others use either no concurrent scheduler, or an exact priority queue.

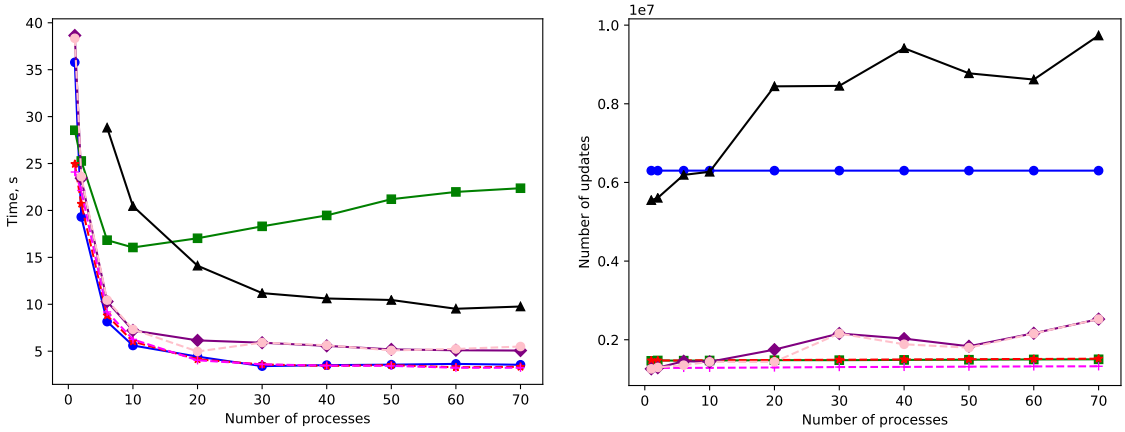
Whenever we have omitted algorithms from the plots or display incomplete data, this indicates poor performance for that algorithm on the metric displayed on the graph: either the algorithm did not converge or the values exceed the limit of the plot.

Tree model. As one can observe on the time plot (Figure 3a), the three algorithms with the best scaling on the tree instance are the synchronous belief propagation, relaxed residual and the weight decay algorithm. For the relaxed algorithms, this mirrors our theoretical analysis from Section 4: as can be seen from Figure 3b, the



(a) Execution time **(b) Number of updates**

Figure 5: The results of the evaluation of the algorithms on Potts model



(a) Execution time **(b) Number of updates**

Figure 6: The results of the evaluation of the algorithms on decoding LDPC code

relaxation incurs very low overhead in terms of additional updates, while the overhead from parallelization is also low. By contrast, the exact residual belief propagation performs exactly the minimum number of updates needed, but scales very badly due to the contention on the priority queue.

We note that on the tree instance, the synchronous belief propagation also scales very well when parallelized. The amount of work can be split evenly between the threads, and only $O(\log n)$ synchronous rounds are required for convergence.

Ising and Potts model. Ising and Potts models represent more challenging instances with lots of cycles, and are generally thought to be more representative of hard general graph instances for belief propagation. As can be seen in Figures 4a and 5a, relaxed algorithms perform consistently well on these instances, with relaxed residual belief propagation giving consistently the fastest convergence. These are followed by the exact splash algorithms, which generally perform slightly worse; however, the scaling seems to be somewhat sensitive to the choice of the parameter H. Both the synchronous and exact residual belief propagation are omitted, as the former did not consistently converge, and the latter was very slow.

An interesting insight is that the exact variants of splash and smart splash do not converge at all in single-threaded executions for some values of the parameter H, but always converge on two and more threads. Similarly, synchronous belief propagation, which has a fixed schedule, does not converge. By contrast, relaxed smart splash converged under all parameter values. We conjecture that

this is due to the phenomenon observed by [22]: exact priority-based algorithms may get stuck in non-convergent cyclic schedules, and injecting randomness into the schedule may help the algorithm to 'escape' these situations. In particular, relaxation to the priority queue, i.e., sometimes executing low-priority items, can provide a such source of randomness. Similarly, an increase in the number of threads leads to the relaxation of the algorithm even for exact schedulers, as several messages are processed in parallel, not only the best one. Thus, we empirically observe that the randomness in the relaxation might help belief propagation to avoid bad cyclic schedules and, therefore, converge.

LDPC model. There are five algorithms that perform similarly (Figure 6a): synchronous belief propagation, relaxed residual belief propagation, the weight decay algorithm, relaxed smart splash with $H = 2$ and, finally, smart splash with $H = 2$. The other algorithms did not converge within our five minutes time limit per experiment.

We note that synchronous belief propagation performs very well on this instance. This is not surprising, as standard belief propagation is known to perform well in LDPC decoding. Generally speaking, the necessary propagation chains seem to be very short on LDPC instances, and the synchronous algorithm parallelizes well in such cases.

5.5 Results: the effects of relaxation

In Table 1 we measure how many more updates the relaxed residual algorithm needs to perform in comparison to the number of updates performed by the standard sequential residual algorithm, denoted as "baseline". We count the total number of updates only approximately: we check the convergence condition only after every 1000 iterations.

The left column indicates whether it is a baseline algorithm or the number of threads for relaxed residual belief propagation. The other columns present the numbers for each model we consider. Each cell contains the corresponding number of updates and how many more updates the relaxed version of the algorithm executed (percentage).

On one process, relaxed residual performs more updates than the baseline does, except in the case of the Potts model. It is expected since our algorithm uses relaxed Multiqueue instead of the strict priority queue. Moreover, as expected the overhead on the number of updates in comparison to the baseline increases with the number of threads. This is again due to the relaxation of the priority queue– recall that we allocate 4× more queues than threads. Interestingly, this overhead is limited even on 70 threads—its maximum value is 9% maximum. This explains the good performance of our algorithm: we reduce the contention by relaxing accesses to the priority queue, while at the same time the total number of updates does not increase significantly.

5.6 Relaxed versus Non-Relaxed Algorithms

In Table 2, we analyze the speedups obtained by the relaxed residual algorithm relative to the best-performing non-relaxed alternative across models and thread counts. We notice that our algorithm outperforms the alternatives in most of the cases, often by a large margin—the highest speedup is of 2.85×, whereas the highest slowdown is of $0.47x$. Both occur on the Potts model, which is generally the most difficult instance in our tests. Overall, the combination of our relaxed scheduling framework combined with the standard residual belief propagation is clearly the algorithm of choice at high thread counts, where it consistently outperforms the alternatives; on the other hand, relaxed residual also performs reasonably well on a single thread, making it a consistently good choice all across the board.

Table 1: Number of additional message updates performed by relaxed residual belief propagation compared to exact residual belief propagation.

		Message updates			
	Threads	Tree	Ising	Potts	LDPC
Exact	1	1000000	2279000	2700000	1464000
Relaxed	1	+0.14%	+0.11%	-0.01%	+0.55%
	2	+0.26%	+0.24%	+0.37%	+0.57%
	6	+0.56%	+2.50%	+2.70%	+0.64%
	10	+0.92%	+3.71%	+4.45%	+1.05%
	20	+2.08%	+5.27%	+5.87%	+1.41%
	30	+2.90%	+6.10%	+6.56%	+1.87%
	40	+3.48%	+6.52%	+7.39%	+2.35%
	50	+5.04%	+6.83%	+7.92%	+2.83%
	60	+4.96%	+7.39%	+8.28%	+3.20%
	70	+5.74%	+7.71%	+8.53%	+3.70%

Table 2: Speedup of relaxed residual belief propagation versus the best non-relaxed alternative on different thread counts. We note that overhead of parallelization can overcome the benefits on small thread counts, as seen in the scaling experiments.

	Speedup			
Threads	Tree	Ising	Potts	LDPC
1	0.89x	**1.08x**	**1.04x**	**1.14x**
2	0.75x	0.51x	0.47x	**1.13x**
6	**1.20x**	0.77x	0.73x	**1.17x**
10	**1.16x**	**1.01x**	0.94x	**1.20x**
20	**1.36x**	**1.66x**	**1.89x**	**1.49x**
30	**1.38x**	**1.88x**	**1.82x**	**1.65x**
40	**1.61x**	**2.21x**	**1.90x**	**1.62x**
50	**1.91x**	**2.67x**	**2.36x**	**1.48x**
60	**1.89x**	**2.66x**	**2.85x**	**1.55x**
70	**1.61x**	**2.71x**	**2.44x**	**1.52x**

6 DISCUSSION

We have investigated the use of relaxed schedulers in the context of the classic belief propagation algorithm for inference on graphical model, and have shown that this approach leads to an efficient family of algorithms, which significantly improve upon the previous state-of-the-art parallelization approaches, by up to 2.8x in our experiments. Overall, our relaxed residual belief propagation has

state-of-the-art performance in both single- and multithreaded regimes, while also being relatively simple to implement, making it a good generic choice for any belief propagation task.

We have also attempted to bound the additional work generated by these schedules in the context of BP. We found that, for well-behaved instances, this can be provably small, although it can also become quite significant in adversarial instances. The experimental results showed surprisingly good performance for relaxed algorithms in practice, and highlighted the intriguing property that they appear to help the algorithm escape cyclic update schedules. One direction for future work would be to better understand this phenomenon, and to attempt to provide stronger bounds for the convergence and work of parallel BP under relaxed schedulers. Another direction is to extend our empirical study to the massively-parallel, multi-machine setting.

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A OPTIMAL SCHEDULE ON TREES

On trees, the belief propagation gives exact marginals under any schedule that updates each edge infinitely often. However, there is an optimal schedule that updates each message exactly once, requiring $O(n)$ message updates [30]. Assume the tree has a fixed root v:

(1) In the first phase, all messages towards the root are updated starting from the leaves; each message is updated only after all its predecessors have been updated.
(2) In the second phase, all messages away from the root are update starting from the root.

This schedule can be modeled in the priority-based scheduling framework as follows:

(1) Initially, the outgoing messages at leaf nodes have priority n, and all other messages have priority 0.
(2) When message is updated with non-zero priority, its priority is changed to 0.
(3) Once all messages $\mu_{k\to i}$ for $k \in N(i)\setminus\{j\}$ have been updated once with non-zero priority, the message $\mu_{i\to j}$ changes to priority to minimum of update priorities of the incoming edges minus one.

This priority function can clearly be implemented by keeping a constant amount of extra information per message. When the above schedule is executed with an exact scheduler, the algorithm will update each message once with non-zero priority before considering any messages with zero priority, and by following the analysis of [30], one can see that the algorithm has converged at that point.

Similarly, in the relaxed version of the schedule, the algorithm has converged once all messages have been updated once with non-zero priority. In addition, some messages may be updated multiple times with priority 0; we call these *wasted* updates, and the updates done while the message has non-zero priority *useful* updates.

Claim 2. *The relaxed version of the optimal schedule on trees performs $O(n + k^2H)$ message updates, where H is the height of the tree.*

Proof. For the purposes of analysis, assign messages into buckets $B_1, B_2 \ldots, B_n$ so that bucket B_ℓ contains the messages that will have their useful update done with priority ℓ. One can observe that the update priority of message $\mu_{i\to j}$ is the $n - d$, where d is the maximum distance from node i to a leaf using a path that does not cross edge $\{i, j\}$. Since this is bounded by the diameter of the tree, there are at most $2H$ non-empty buckets.

Assume that all messages in buckets $B_n, B_{n-1}, \ldots, B_{\ell+1}$ have been already had a useful update. We now show that in there can be at most k^2 wasted updates before all messages in B_ℓ have had a useful update. Since all earlier buckets have been processed, all messages in B_ℓ have either already had a useful update, or have priority ℓ. Let b be the number of messages remaining in bucket B_ℓ:

- While $b \geq k$, there are at least k messages with priority ℓ, so each update is a useful update of a message in B_ℓ
- When $b < k$, there can be wasted updates. However, since buckets $B_n, B_{n-1}, \ldots, B_{\ell+1}$ have had all useful updates, the top elements in the schedule will be from bucket B_ℓ, and thus by the guarantees of the scheduler, there can be at most $k - 1$ wasted updates before the top element is processed. Thus, in $b(k-1) = O(k^2)$ updates, all remaining messages of B_ℓ will have their useful update.

By an inductive argument, all non-empty buckets have been processed after $O(k^2H)$ wasted updates, so the total number of updates is $O(n + k^2H)$. □

B EXECUTION TIME OF RANDOM SYNCHRONOUS ALGORITHM

In Table 3 we present the execution time of random synchronous algorithm on 70 threads (RS 70) with different values of $lowP = 0.1, 0.4$ and 0.7, where the parameter $lowP$ controls the random selection fraction p in steps where the algorithm is converging slowly (see Section 2.4). We compare it with the execution time of two baselines: Synchronous algorithm on 70 threads (S 70) and Relaxed Residual on one process (RR 1). Cells with '—' indicate executions that either take more than five minutes to run or simply do not converge.

To summarize, we did not include the execution time of random synchronous algorithm in the scaling plots since it exceeds the execution time of one of the baselines in all cases.

Table 3: Randomized synchronous algorithm versus baselines.

Algorithm		Running time (s) Tree	Ising	Potts	LDPC
S 70		4.088	—	—	3.504
RR 1		5.579	9.012	10.583	25.663
RS 70	$lowP = 0.1$	37.052	62.629	—	28.543
	$lowP = 0.4$	8.420	20.396	—	7.269
	$lowP = 0.7$	6.306	12.581	—	4.791