Assignment#4 Sample Key
1. NotDominating(ND) = \{ f \mid \text{for some } x, f(x) < x \}.

a.) Show some minimal quantification of some known primitive recursive predicate that provides an upper bound for the complexity of ND.

\[ \exists <x,t> [\text{STP}(f,x,t) \& (\text{VALUE}(f,x,t) < x)] \]

b.) Use Rice’s Theorem to prove that ND is undecidable. Be Complete.

ND is non-trivial as \(C_0(x) = 0 \in \text{ND}\) and \(S(x) = x+1 \notin \text{ND}\)

Let \(f, g\) be two arbitrary indices of procedures such that \(\forall x f(x) = g(x)\)

\(f \in \text{ND} \iff \exists x f(x) < x\) iff \(f(x_0) < x_0\) for some \(x_0\) iff \(g(x_0) < x_0\) as \(\forall x f(x) = g(x)\) implies \(\exists x g(x) < x\) iff \(g \in \text{ND}\)

\(f \notin \text{ND} \iff \forall x [f(x) \downarrow \text{implies } f(x) > x]\) iff \(\forall x [g(x) \downarrow \text{implies } g(x) > x]\) as \(\forall x f(x) = g(x)\) iff \(g \notin \text{ND}\)

c.) Show that \(K = \{ f \mid f(f) \text{ converges} \}\) is many-one reducible to ND.

Let \(f\) be an arbitrary index. From \(f\), define \(\forall x F_f(x) = f(f) - f(f)\).

\(f \in K\) implies \(\forall x F_f(x) = 0\) implies \(F_f \in \text{ND}\).

\(f \notin K\) implies \(\forall x F_f(x) \text{ diverges} \implies F_f \notin \text{ND}\).

Thus, \(K \leq_m \text{ND}\)

d.) Show that ND is many-one reducible to \(K = \{ f \mid f(f) \text{ converges} \}\)

Let \(f\) be an arbitrary index. From \(f\), define \(\forall y F_f(y) = \exists <x,t> [\text{STP}(f,x,t) \& (\text{VALUE}(f,x,t) < x)]\)

\(f \in \text{ND}\) implies \(\forall y F_f(y) \text{ converges} \implies F_f(F_f) \text{ converges} \implies F_f \in K\)

\(f \notin \text{ND}\) implies \(\forall y F_f(y) \text{ diverges} \implies F_f \notin K\)

Thus, \(\text{ND} \leq_m K\)
2. **AlwaysDominates** (AD) = \{ f | \text{for all } x, f(x) > x \}

a.) Show some minimal quantification of some known primitive recursive predicate that provides an upper bound for the complexity of AD.

\[ \forall x \exists t \ (\text{STP}(f,x,t) \& (\text{VALUE}(f,x,t) > x)) \]

b.) Use Rice’s Theorem to prove that AD is undecidable. Be Complete.

AD is non-trivial as \( S(x) = x+1 \in \text{AD} \) and \( C_0(x) = 0 \notin \text{AD} \)

Let \( f,g \) be two arbitrary indices of procedures such that \( \forall x f(x) = g(x) \)

\( f \in \text{AD} \iff \forall x f(x) < x \iff \forall x g(x) < x \iff g \in \text{ND} \)

c.) Show that \( \text{TOT} = \{ f | \text{for all } x, f(x) \text{ converges} \} \) is many-one reducible to AD.

Let \( f \) be an arbitrary index. From \( f \), define \( \forall x F_f(x) = f(x)-f(x) + x + 1. \)

\( f \in \text{TOT} \) implies \( \forall x F_f(x) = x+1 \) implies \( F_f \in \text{AD}. \)
\( f \notin \text{TOT} \) implies \( \exists x F_f(x) \) diverges implies \( F_f \notin \text{AD}. \)

Thus, \( \text{TOT} \leq_m \text{AD} \)

d.) Show that AD is many-one reducible to \( \text{TOT} = \{ f | \text{for all } x, f(x) \text{ converges} \} \)

Let \( f \) be an arbitrary index. From \( f \), define \( \forall x F_f(x) = \mu y \ [ f(x) > x ] \)

\( f \in \text{AD} \) implies \( \forall x F_f(x) \) converges implies \( F_f \in \text{TOT} \)
\( f \notin \text{AD} \) implies \( \exists x F_f(x) \) diverges implies \( F_f \notin \text{TOT}. \)

Thus, \( \text{AD} \leq_m \text{TOT} \)