Assignment#4 Key
1. HasDup(ND) = { f | for some x, y, x \neq y, f(x) = f(y) }

a.) Show some minimal quantification of some known primitive recursive predicate that provides an upper bound for the complexity of ND.

∀<x,y,t> [STP(f,x,t) & STP(f,y,t) & x \neq y & (VALUE(f,x,t) = VALUE(f,xyt))]

b.) Use Rice’s Theorem to prove that ND is undecidable. Be Complete.

ND is non-trivial as \( C0(x) = 0 \in ND \) and \( S(x) = x+1 \notin ND \)

Let \( f, g \) be two arbitrary indices of procedures such that \( \forall x \ f(x) = g(x) \)

\( f \in ND \) iff \( \exists x, y \ ((x \neq y) \land f(x) = f(y)) \) iff \( f(x_0) = f(y_0) \) for some \( x_0, y_0 \) iff \( g(x_0) = g(y_0) \) as \( \forall x \ f(x) = g(x) \) implies \( \exists x, y \ ((x \neq y) \land g(x) = g(y)) \) iff \( g \in ND \)

\( f \notin ND \) iff \( \forall x, y \ ((x \neq y) \implies f(x) \neq f(y)) \) iff \( \forall x, y \ ((x \neq y) \implies g(x) \neq g(y)) \) as \( \forall x \ f(x) = g(x) \) iff \( g \notin ND \)

c.) Show that \( K = \{ f | \text{f(f) converges} \} \) is many-one reducible to ND.

Let \( f \) be an arbitrary index. From \( f \), define \( \forall x \ F_f(x) = f(f) - f(f) \).

\( f \in K \) implies \( \forall x \ F_f(x) = 0 \) implies \( F_f \in ND \).

\( f \notin K \) implies \( \forall x \ F_f(x) \uparrow \) implies \( F_f \notin ND \).

Thus, \( K \leq_m ND \)

d.) Show that ND is many-one reducible to \( K = \{ f | \text{f(f) converges} \} \)

Let \( f \) be an arbitrary index. From \( f \), define \( \forall y F_f(y) = \exists<x,y,t> \ [STP(f,x,t) \land STP(f,y,t) \land x \neq y \land (\text{VALUE}(f,x,t) = \text{VALUE}(f,xyt))] \)

\( f \in ND \) implies \( \forall y F_f(y) \downarrow \) implies \( F_f(F_f) \) converges implies \( F_f \in K \)

\( f \notin ND \) implies \( \forall y F_f(y) \uparrow \) implies \( F_f \notin K \).

Thus, \( ND \leq_m K \)
2. \textbf{AlwaysDominates(AD)} = \{ f \mid \text{for all } x, f(x) > x \} \\

a.) Show some minimal quantification of some known primitive recursive predicate that provides an upper bound for the complexity of AD.
\[ \forall x \exists t \left[ \text{STP}(f,x,t) \land (\text{VALUE}(f,x,t) > x) \right] \]

b.) Use Rice's Theorem to prove that AD is undecidable. Be Complete.
AD is non-trivial as \( S(x) = x+1 \in \text{AD} \) and \( C_0(x) = 0 \notin \text{AD} \)

Let \( f,g \) be two arbitrary indices of procedures such that \( \forall x \, f(x) = g(x) \)
\[ f \in \text{AD} \iff \forall x \, f(x) > x \iff \forall x \, g(x) > x \iff g \in \text{ND} \]

c.) Show that \( \text{TOT} = \{ f \mid \text{for all } x, f(x) \text{ converges} \} \) is many-one reducible to AD.
Let \( f \) be an arbitrary index. From \( f \), define \( \forall x \, F_f(x) = f(x) - f(x) + x + 1 \).
\[ f \in \text{TOT} \iff \forall x \, F_f(x) = x + 1 \iff F_f \in \text{AD} \]
\[ f \notin \text{TOT} \iff \exists x \, F_f(x) \downarrow \iff F_f \notin \text{AD} \]

Thus, \( \text{TOT} \leq_m \text{AD} \)

d.) Show that AD is many-one reducible to \( \text{TOT} = \{ f \mid \text{for all } x, f(x) \text{ converges} \} \)
Let \( f \) be an arbitrary index. From \( f \), define \( \forall x \, F_f(x) = \exists y \left[ f(x) > x \right] \)
\[ f \in \text{AD} \implies \forall x \, F_f(x) \downarrow \iff F_f \in \text{TOT} \]
\[ f \notin \text{AD} \implies \exists x \, F_f(x) \uparrow \iff F_f \notin \text{TOT} \]

Thus, \( \text{AD} \leq_m \text{TOT} \)