Computability & Complexity Theory

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Review of Automata and Formal Languages

Finite State Automata

Concrete Model of FSA

L is a finite state (regular) language over finite alphabet Σ
Each x_i is a character in Σ
$w = x_1\ x_2 \ldots x_n$ is a string to be tested for membership in L

- Arrow above represents read head that starts on left.
- $q_0 \in Q$ (finite state set) is initial state of machine.
- Only action at each step is to change state based on character being read and current state. State change is determined by a transition function $\delta: Q \times \Sigma \rightarrow Q$.
- Once state is changed, read head moves right.
- Machine stops when head passes last input character.
- Machine accepts string as member of L if it ends up in a state from Final State set $F \subseteq Q$.

Finite State Automata

- A deterministic finite state automaton (DFA) A is defined by a 5-tuple
  $A = (Q,\Sigma,\delta,q_0,F)$, where
  - Q is a finite set of symbols called the states of A
  - Σ is a finite set of symbols called the alphabet of A
  - δ is a function from $Q \times \Sigma$ into Q ($\delta: Q \times \Sigma \rightarrow Q$) called the transition function of A
  - $q_0 \in Q$ is a unique element of Q called the start state
  - F is a subset of Q ($F \subseteq Q$) called the final states (can be empty)

DFA Transitions

- Given a DFA, $A = (Q,\Sigma,\delta,q_0,F)$, we can definition the reflexive transitive closure of δ, $\delta^*:Q \times \Sigma^* \rightarrow Q$, by
  - $\delta^*(q,\lambda) = q$ where λ is the string of length 0
    - Note that text uses ϵ rather than λ as symbol for string of length zero
  - $\delta^*(q,ax) = \delta^*(\delta(q,a),x)$, where $a \in \Sigma$ and $x \in \Sigma^*$
  - Note that this means
    $\delta^*(q,a) = \delta(q,a)$, where $a \in \Sigma$ as $a = a\lambda$
- We also define the transitive closure of δ, δ^+, by
  - $\delta^+(q,w) = \delta^*(q,w)$ when $|w|>0$ or, equivalently, $w \in \Sigma^+$
- The function δ^* describes every step of computation by the automaton starting in some state until it runs out of characters to read

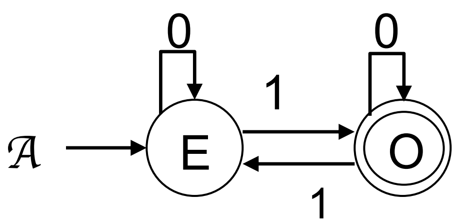
Regular Languages and DFAs

- Given a DFA, $A = (Q,\Sigma,\delta,q_0,F)$, we can define the language accepted by A as those strings that cause it to end up in a final state once it has consumed the entire string
- Formally, the language accepted by A is
  - $\{ w \mid \delta^*(q_0,w) \in F \}$
- We generally refer to this language as *L*(A)
- We define the notion of a Regular Language by saying that a language is Regular if and only if it is accepted (recognized) by some DFA

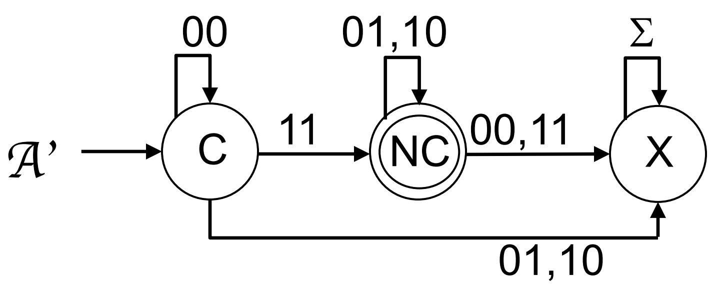
State Diagram

- A finite state automaton can be described by a state diagram, where
  - Each state is represented by a node labelled with that state, e.g., q
  - The state state has an arc entering it with no source, e.g., $\rightarrow q_0$
  - Each transition δ(q,a) = s is represented by a directed arc from node q to node s that is labelled with the letter a, e.g., $q \xrightarrow{a} s$
  - Each final state has an extra circle around its node, e.g., f

Sample DFAs # 1, 2

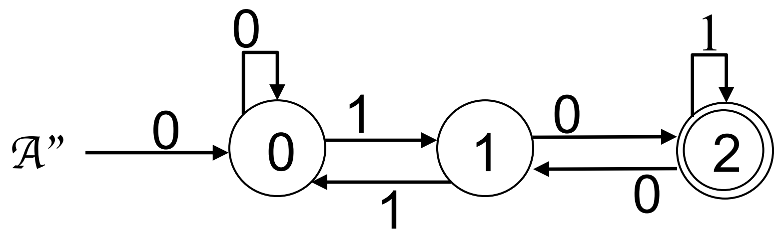


$\mathcal{A}$ = ({E,O}, {0,1}, δ, E, {O}), where δ is defined by above diagram. L($\mathcal{A}$) = { w | w is a binary string of odd parity }



$\mathcal{A}$' = ({C,NC,X}, {00,01,10,11}, δ', C, {NC}), where δ' is defined by above diagram. L($\mathcal{A}$') = { w | w is a pair of binary strings where the bottom string is the 2's complement of the top one, both read least (lsb) to most significant bit (msb) }

Sample DFA # 3



$\mathcal{A}$” = ({0,1,2}, {0,1}, δ”, 0, {2}), where δ” is defined by above diagram. L($\mathcal{A}$”) = { w | w is a binary string of length at least 1 being read left to right (msb to lsb) that, when interpreted as a decimal number divided by 3, has a remainder of 2 }

State Transition Table

- A finite state automaton can be described by a state transition table with $|Q|$ rows and $|\Sigma|$ columns
- Rows are labelled with state names and columns with input letters
- The start state has some indicator, e.g., a greater than sign ($>q$) and each final state has some indicator, e.g., an underscore ($\underline{f}$)
- The entry in row q, column a, contains $\delta(q,a)$
- In general we will use state diagrams, but transition tables are useful in some cases (state minimization)

Sample DFA # 4

		0	**1**
	0 % 5	0 % 5	1 % 5
	1 % 5	2 % 5	3 % 5
	2 % 5	4 % 5	0 % 5
Accept State	**3 % 5**	1 % 5	2 % 5
	4 % 5	3 % 5	4 % 5

$\mathcal{A}'''$ = ({0%5,1%5,2%5,3%5,4%5}, {0,1}, δ''', 0, {3%5}), where δ''' is defined by above diagram.
L($\mathcal{A}''$) = { w | w is a binary string of length at least 1 being read left to right (msb to lsb) that, when interpreted as a decimal number divided by 5, has a remainder of 3 }

Really, this is better done as a state diagram, but have put this up so you can see the pattern.

Sample DFA # 5

	A-Z	a-z	0-9	@#$%^&
⇨ **Empty**	A	a	0	@
A	A	Aa	A0	A@
a	Aa	a	a0	a@
0	A0	a0	0	0@
@	A@	a@	0@	@
Aa	Aa	Aa	Aa0	Aa@
A0	A0	Aa0	A0	A0@
A@	A@	Aa@	A0@	A@
a0	Aa0	a0	a0	a0@
a@	Aa@	a@	a0@	a@
0@	A0@	a0@	0@	0@
Aa0	Aa0	Aa0	Aa0	Aa0@
Aa@	Aa@	Aa@	Aa0@	Aa@
A0@	A0@	Aa0@	A0@	A0@
a0@	Aa0@	a0@	a0@	a0@
Aa0@	Aa0@	Aa0@	Aa0@	Aa0@

This checks a string to see if it's a legal password. In our case, a legal password must contain at least one of each of the following: lower case letter, upper case letter, number, and special character from the following set {!@#$%^&}. No other characters are allowed

DFA Closure

- Regular languages (those recognized by DFAs) are closed under complement, union, intersection, difference and exclusive or ($\oplus$) and many other set operations
- Let $A_1 = (Q_1,\Sigma,\delta_1,q_0,F_1)$, $A_2 = (Q_2,\Sigma,\delta_2,s_0,F_2)$ be arbitrary DFAs
- $\Sigma^*\text{-}L(A_1)$ is recognized by $A_1^C = (Q_1,\Sigma,\delta_1,q_0,Q_1\text{-}F_1)$
- Define $A_3 = (Q_1 \times Q_2,\Sigma,\delta_3,<q_0,s_0>,F_3)$ where $\delta_3(<q,s>,a)= <\delta_1(q,a),\delta_2(s,a)>$, $q \in Q_1$, $s \in Q_2$, $a \in \Sigma$
  - $L(A_1) \cup L(A_2)$ is recognized when $F_3=(F_1 \times Q_2) \cup (Q_1 \times F_2)$
  - $L(A_1) \cap L(A_2)$ is recognized when $F_3=F_1 \times F_2$
  - $L(A_1)$ - $L(A_2)$ is recognized when $F_3=F_1 \times (Q_2\text{-}F_2)$
  - $L(A_1) \oplus L(A_2)$ is recognized when $F_3=F_1 \times (Q_2\text{-}F_2) \cup (Q_1\text{-}F_1) \times F_2$

Complement of Regular Sets

- Let $A = (Q,\Sigma,\delta,q_0,F)$
- Simply create new automaton
  $A^C = (Q,\Sigma,\delta,q_0,Q\text{-}F)$
- $L(A^C) = \{ w \mid \delta^*(q_0,w) \in Q\text{-}F \} =$
  $\{ w \mid \delta^*(q_0,w) \notin F \} =$
  $\{ w \mid w \notin L(A) \}$
- Again, imagine trying to do this in the context of regular expressions
- Choosing the right representation can make a very big difference in how easy or hard it is to prove some property is true

Parallelizing DFAs

- Regular sets can be shown closed under many binary operations using the notion of parallel machine simulation
- Let $A_1 = (Q_1,\Sigma,\delta_1,q_0,F_1)$ and $A_2 = (Q_2,\Sigma,\delta_2,s_0,F_2)$ where $Q_1 \cap Q_2 = \emptyset$
- $B = (Q_1 \times Q_2,\Sigma,\delta_3,<q_0,s_0>,F_3)$ where $\delta_3(<q,s>,a) = < \delta_1(q,a), \delta_2(s,a) >$
- Union is $F_3 = F_1 \times Q_2 \cup Q_1 \times F_2$
- Intersection is $F_3 = F_1 \times F_2$
  - Can do by combining union and complement
- Difference is $F_3 = F_1 \times (Q_2 - F_2)$
  - Can do by combining intersection and complement
- Exclusive Or is $F_3=F_1 \times (Q_2-F_2) \cup (Q_1-F_1) \times F_2$

Non-determinism NFA

- A non-deterministic finite state automaton (NFA) A is defined by a 5-tuple $A = (Q,\Sigma,\delta,q_0,F)$, where
  - Q is a finite set of symbols called the states of A
  - Σ is a finite set of symbols called the alphabet of A
  - δ is a function from $Q \times \Sigma_e$ into $P(Q) = 2^Q$; Note: $\Sigma_e = (\Sigma \cup \{\lambda\})$ ($\delta: Q \times \Sigma_e \to P(Q)$) called the transition function of A; by definition $q \in \delta(q,\lambda)$
  - $q_0 \in Q$ is a unique element of Q called the start state
  - F is a subset of Q ($F \subseteq Q$) called the final states
  - Note that a state/input (called a discriminant) can lead nowhere new, one place or many places in an NFA; moreover, an NFA can jump between states even without reading any input symbol
  - For simplicity, we often extend the definition of $\delta: Q \times \Sigma_e$ to a variant that handles sets of states, where $\delta: P(Q) \times \Sigma_e$ is defined as $\delta(S,a) = \cup_{q \in S} \delta(q,a)$, where $a \in \Sigma_e$ – if $S=\emptyset$, $\cup_{q \in S} \delta(q,a) = \emptyset$

NFA Transitions

- Given an NFA, $A = (Q,\Sigma,\delta,q_0,F)$, we can define the reflexive transitive closure of δ, $\delta^*:P(Q) \times \Sigma^* \rightarrow P(Q)$, by
  - λ-Closure(S) = $\{ t \mid t \in \delta^*(S,\lambda)\}$, $S \in P(Q)$ – extended δ
  - $\delta^*(S,\lambda) = \lambda$-Closure(S)
  - $\delta^*(S,ax) = \delta^*(\lambda\text{-Closure}(\delta(S,a),x))$, where $a \in \Sigma$ and $x \in \Sigma^*$
    - Note that $\delta^*(S,ax) = \cup_{q\in S} \cup_{p\in\lambda\text{-Closure}(\delta(q,a))} \delta^*(p,x)$, where $a \in \Sigma$ and $x \in \Sigma^*$
- We also define the transitive closure of δ, δ^+, by
  - $\delta^+(S,w) = \delta^*(S,w)$ when $|w|>0$ or, equivalently, $w \in \Sigma^+$
- The function δ^* describes every "possible" step of computation by the non-deterministic automaton starting in some state until it runs out of characters to read

NFA Languages

- Given an NFA, $A = (Q,\Sigma,\delta,q_0,F)$, we can define the language accepted by A as those strings that allow it to end up in a final state once it has consumed the entire string – here we just mean that there is some accepting path
- Formally, the language accepted by A is
  - $\{ w \mid (\delta^*(\lambda\text{-Closure}(\{q_0\}),w) \cap F) \neq \emptyset \}$
- Notice that we accept if there is any set of choices of transitions that lead to a final state

Finite State Diagram

- A non-deterministic finite state automaton can be described by a finite state diagram, except
  - We now can have transitions labelled with λ
  - The same letter can appear on multiple arcs from a state q to multiple distinct destination states

Equivalence of DFA and NFA

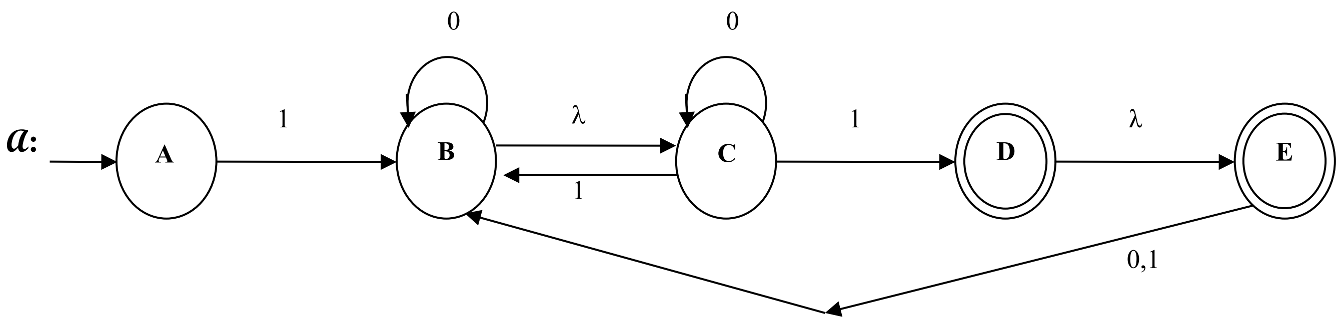
- Clearly every DFA is an NFA except that $\delta(q,a) = s$ becomes $\delta(q,a) = \{s\}$, so any language accepted by a DFA can be accepted by an NFA.
- The challenge is to show every language accepted by an NFA is accepted by an equivalent DFA. That is, if A is an NFA, then we can construct a DFA A', such that $L(A') = L(A)$.

Constructing DFA from NFA

- Let A = (Q,Σ,δ,q_0,F) be an arbitrary NFA
- Let S be an arbitrary subset of Q.
  – Construct the sequence seq(S) to be a sequence that contains all elements of S in lexicographical order, using angle brackets to . That is, if S={q1, q3, q2} then seq(S)=<q1,q2,q3>. If S=Ø then seq(S)=<>
- Our goal is to create a DFA, A', whose state set contains seq(S), whenever there is some w such that $S=\delta^*(q_0,w)$
- To make our life easier, we will act as if the states of A' are sets, knowing that we really are talking about corresponding sequences

λ-Closure

- Define the λ-Closure of a state q as the set of states one can arrive at from q, without reading any additional input.
- Formally $\lambda\text{-Closure}(q) = \{ t \mid t \in \delta^*(q,\lambda) \}$
- We can extend this to $S \in P(Q)$ by
  $\lambda\text{-Closure}(S) = \{ t \mid t \in \delta^*(q,\lambda),\ q \in S \} = \{ t \mid t \in \lambda\text{-Closure}(q), q \in S\}$



State	A	B	C	D	E
λ-closure	{ A }	{ B , C }	{ C }	{ D, E }	{ E }

Details of DFA

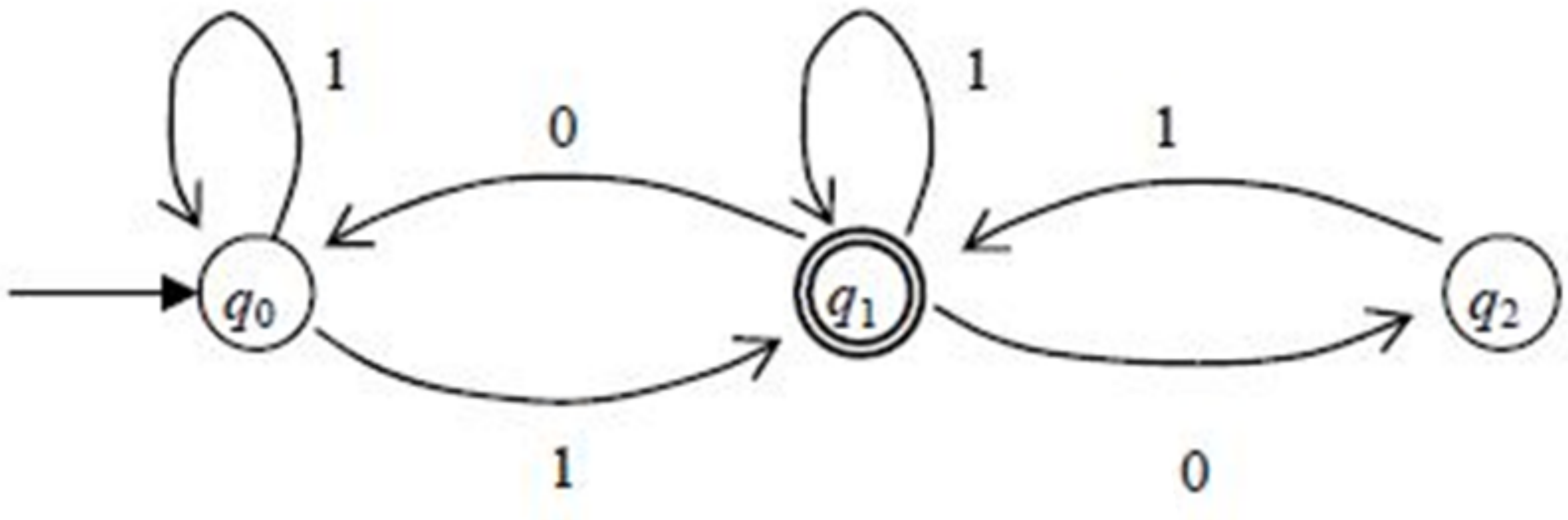
- Let $A = (Q,\Sigma,\delta,q_0,F)$ be an arbitrary NFA
- In an abstract sense,
  $A' = (\langle P(Q)\rangle,\Sigma,\delta', \langle\lambda\text{-Closure}(\{q_0\})\rangle, F')$,
  but we really don't need so many states ($2^{|Q|}$) and we can iteratively determine those needed by starting at λ-Closure($\{q_0\}$) and keeping only states reachable from here
- Define $\delta'(\langle S\rangle,a) = \langle\lambda\text{-Closure}(\delta(S,a))\rangle = \langle\cup_{q\in S}\, \lambda\text{-Closure}(\delta(q,a))\rangle$, where $a\in\Sigma$, $S \in P(Q)$
- $F' = \{\langle S\rangle \in \langle P(Q)\rangle \mid (S \cap F) \neq \emptyset \}$

Regular Languages and NFAs

- Showing that every NFA can be simulated by a DFA that accepts the same language proves the following

- A language is Regular if and only if it is accepted (recognized) by some NFA

Convert from NFA to DFA



Regular Expressions

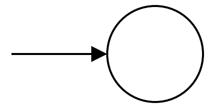
- Primitive:
  - Φ denotes {}
  - λ denotes $\{\lambda\}$
  - a where a is in Σ denotes {a}
- Closure:
  - If R and S are regular expressions then so are $R \cdot S$, $R + S$ and R^*, where
    - $R \cdot S$ denotes $RS = \{ xy \mid x \text{ is in } R \text{ and } y \text{ is in } S \}$
    - $R + S$ denotes $R \cup S = \{ x \mid x \text{ is in } R \text{ or } x \text{ is in } S \}$
    - R^* denotes R^*
- Parentheses are used as needed

Regular Sets = Regular Languages

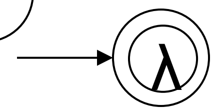
- Show every regular expression denotes a language recognized by a finite state automaton (can do deterministic or non-deterministic)
- Show every Finite State Automata recognizes a language denoted by a regular expression

Every Regular Set is a Regular Language

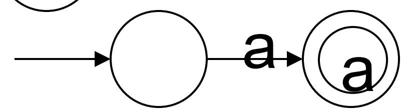
- Primitive:
  - Φ denotes {}



  - λ denotes {λ}



  - a where a is in Σ denotes {a}

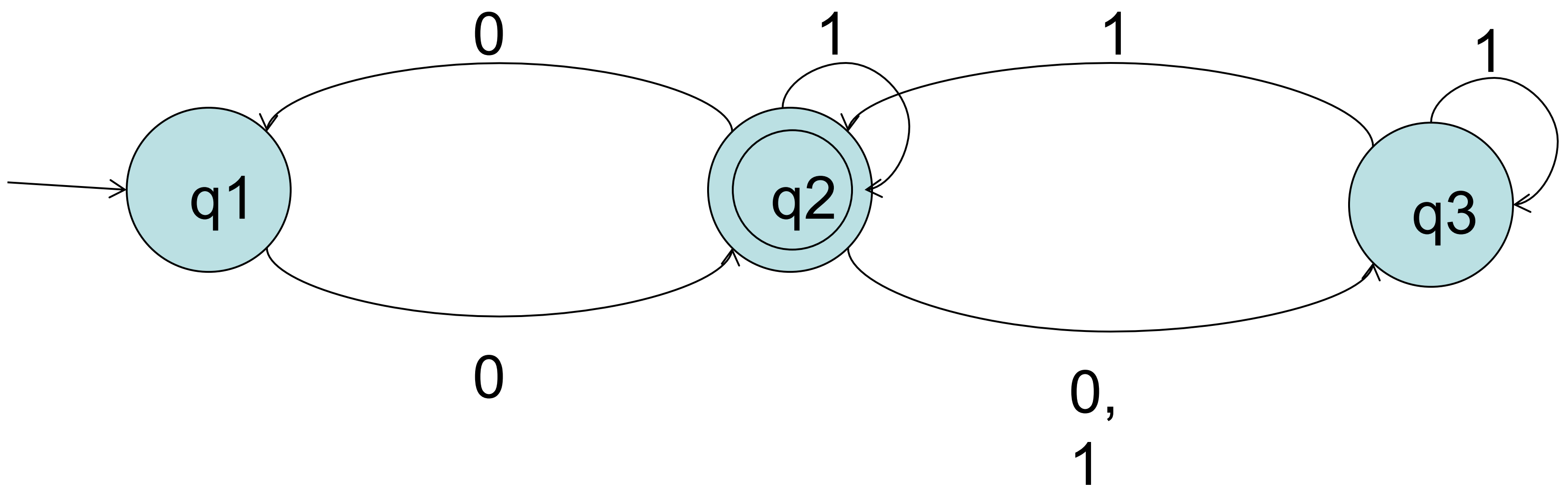


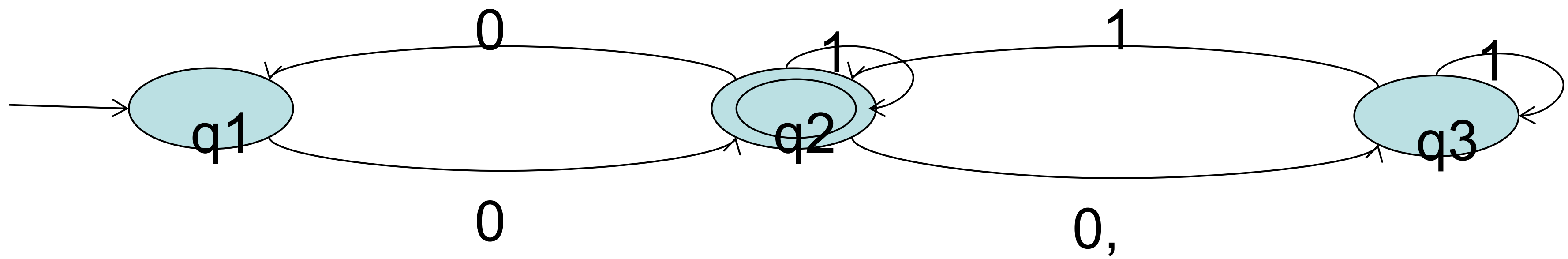
- Closure: (Assume that R's and S's states do not overlap)
  - R · S start with machine for R, add λ transitions from every final state of R's recognizer to start state of S, making final state of S final states of new machine
  - R + S create new start state and add λ transitions from new state to start states of each of R and S, making union of R's and S's final states the new final states
  - R* add λ transitions from each final state of R back to its start state, keeping original start and final states (gets R^+) – FIX?

Every Regular Language is a Regular Set Using R_{ij}^k

- This is a challenge that can be addressed in multiple ways but I like to start with the R_{ij}^k approach. Here's how it works.
- Let $A = (Q,\Sigma,\delta,q_1,F)$ be a DFA, where $Q = \{q_1,q_2, \ldots , q_n\}$
- $R_{ij}^k = \{w \mid \delta^*(q_i,w) = q_j$, and no intermediate state visited between q_i and q_j, while reading w, has index > k
- Basis: $k=0$, $R_{ij}^0 = \{ a \mid \delta(q_i,a) = q_j \}$ sets are either Φ, λ, or an element of Σ or λ + element of Σ, and so are regular sets
- Inductive hypothesis: Assume R_{ij}^m are regular sets for $0 \le m \le k$
- Inductive step: $k+1$, $R_{ij}^{k+1} = (R_{ij}^k + R_{ik+1}^k \cdot (R_{k+1k+1}^k)^* \cdot R_{k+1j}^k)$
- $L(A) = +_{f\in F} R_{1f}^n$

Convert to RE





- $R_{11}^0 = \lambda$ $\quad$ $R_{12}^0 = 0$ $\quad$ $R_{13}^0 = \phi$
- $R_{21}^0 = 0$ $\quad$ $R_{22}^0 = \lambda + 1$ $\quad$ $R_{23}^0 = 0 + 1$
- $R_{31}^0 = \phi$ $\quad$ $R_{32}^0 = 1$ $\quad$ $R_{33}^0 = \lambda + 1$
- $R_{11}^1 = \lambda$ $\quad$ $R_{12}^1 = 0$ $\quad$ $R_{13}^1 = \phi$
- $R_{21}^1 = 0$ $\quad$ $R_{22}^1 = \lambda + 1 + 00$ $\quad$ $R_{23}^1 = 0 + 1$
- $R_{31}^1 = \phi$ $\quad$ $R_{32}^1 = 1$ $\quad$ $R_{33}^1 = \lambda + 1$
- $R_{11}^2 = \lambda + 0(1+00)^*0$ $\quad$ $R_{12}^2 = 0(1+00)^*$ $\quad$ $R_{13}^2 = 0(1+00)^*(0+1)$
- $R_{21}^2 = (1+00)^*0$ $\quad$ $R_{22}^2 = (1+00)^*$ $\quad$ $R_{23}^2 = (1+00)^*(0+1)$
- $R_{31}^2 = 1(1+00)^*0$ $\quad$ $R_{32}^2 = 1(1+00)^*$ $\quad$ $R_{33}^2 = \lambda+1+1(1+00)^*(0+1)$
- $L = R_{12}^3 =$
  $0(1+00)^* + 0(1+00)^*(0+1)\ (1+1(1+00)^*(0+1))^*\ 1(1+00)^*$

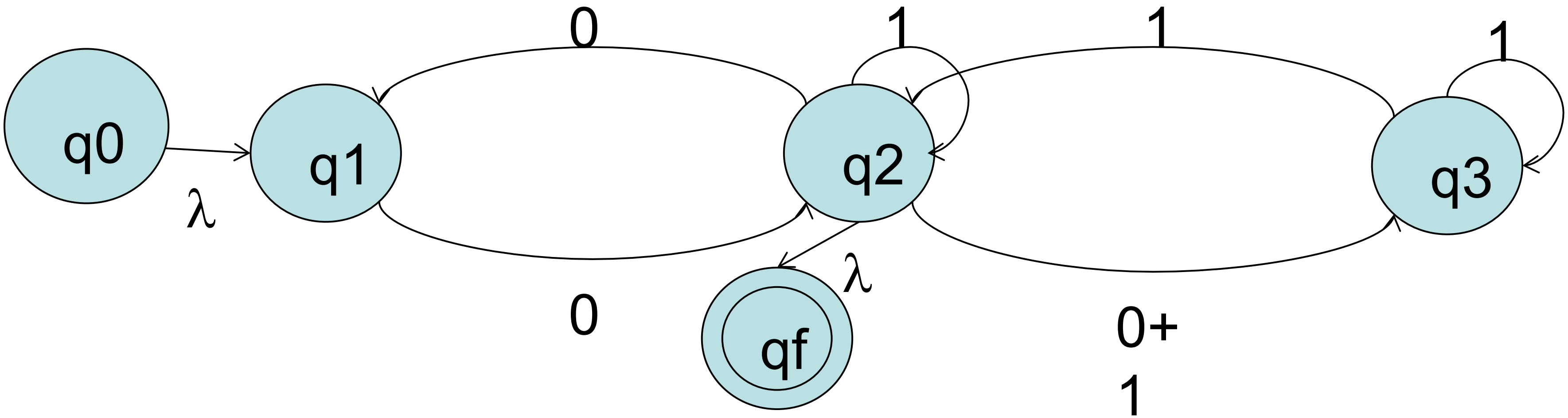
State Ripping Concept

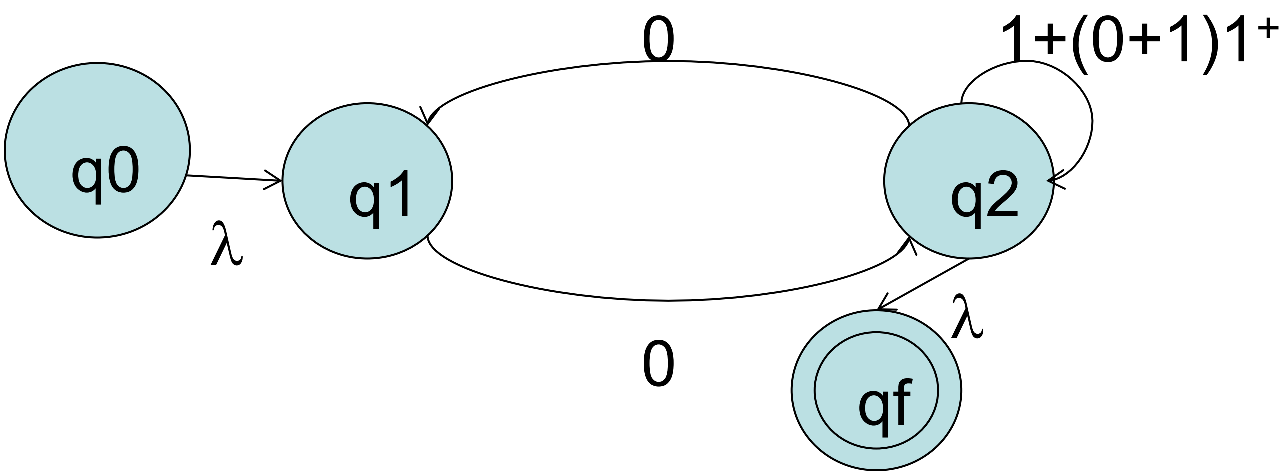
- This has its motivation from R_{ij}^k approach
- Add a new start state and add a λ–transition to existing start state
- Add a new final state q_f and insert λ–transitions from all existing final states to the new one; make the old final states non-final
- Leaving the start and final states, successively pick states to remove
- For each state to be removed, change the arcs of every pair of externally entering and exiting arcs to reflect the regular expression that describes all strings that could result is such a double transition; be sure to account for loops in the state being removed. Also, or (+) together expressions that have the same start and end nodes
- When have just start and final, the regular expression that leads from start to final describes the associated regular set

State Ripping Details

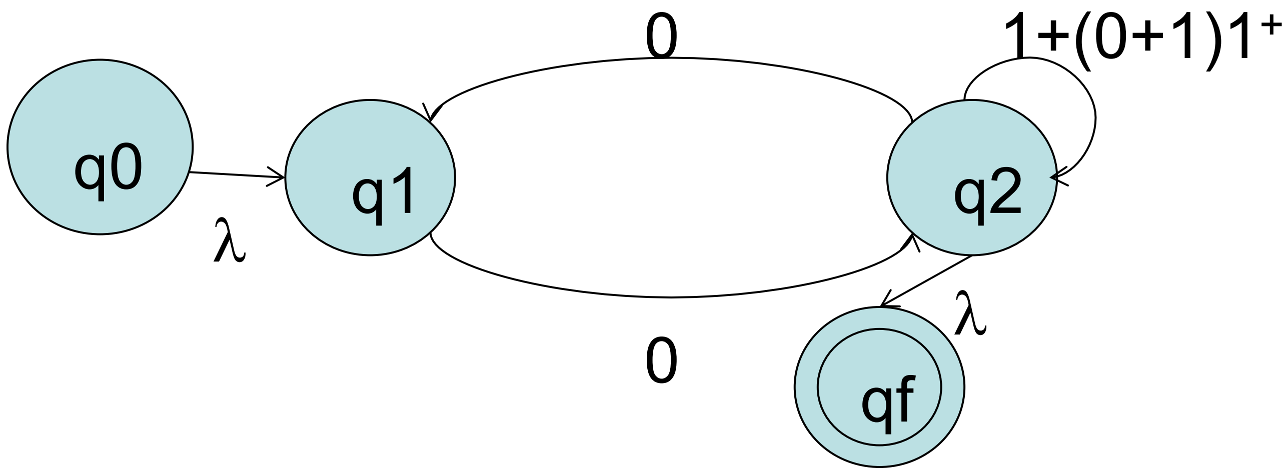
- Let B be the node to be removed
- Let e1 be the regular expression on the arc from some node A to some node B (A≠B); e2 be the expression from B back to B (or λ if there is no recursive arc); e3 be the expression on the arc from B to some other node C (C ≠B but C could be A); e4 be the expression from A to C
- Erase the existing arcs from A to B and A to C, adding a new arc from A to C labelled with the expression
  e4 + e1 e2* e3
- Do this for all nodes that have edges to B until B has no more entering edges; at this point remove B and any edges it has to other nodes and itself
- Iterate until all but the start and final nodes remain
- The expression from start to final describes regular set that is equivalent to regular language accepted by original automaton
- Note: Your choices of the order of removal make a big difference in how hard or easy this is

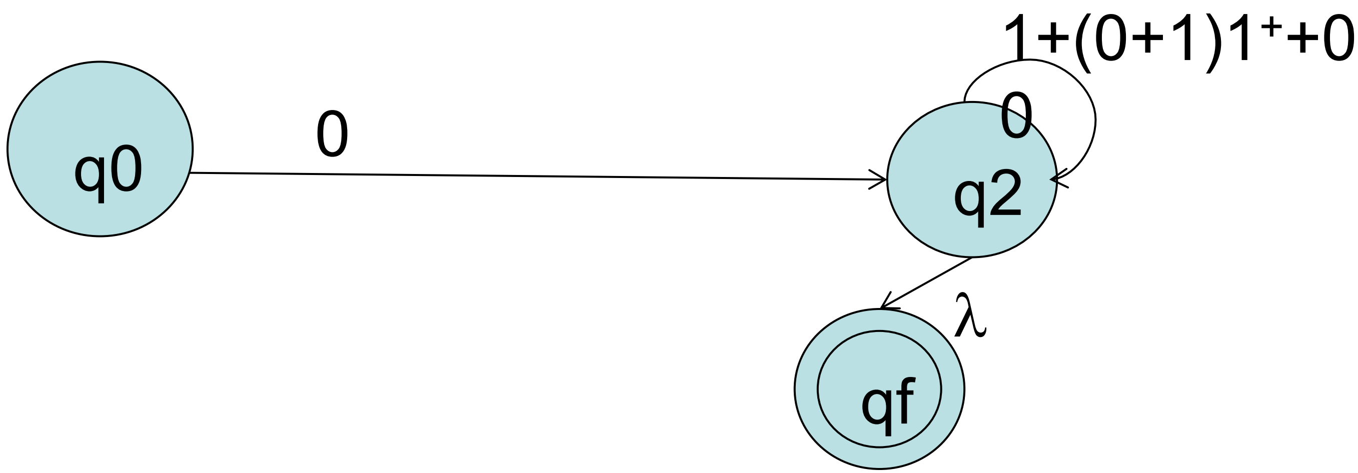
Use Ripping; Rip q3



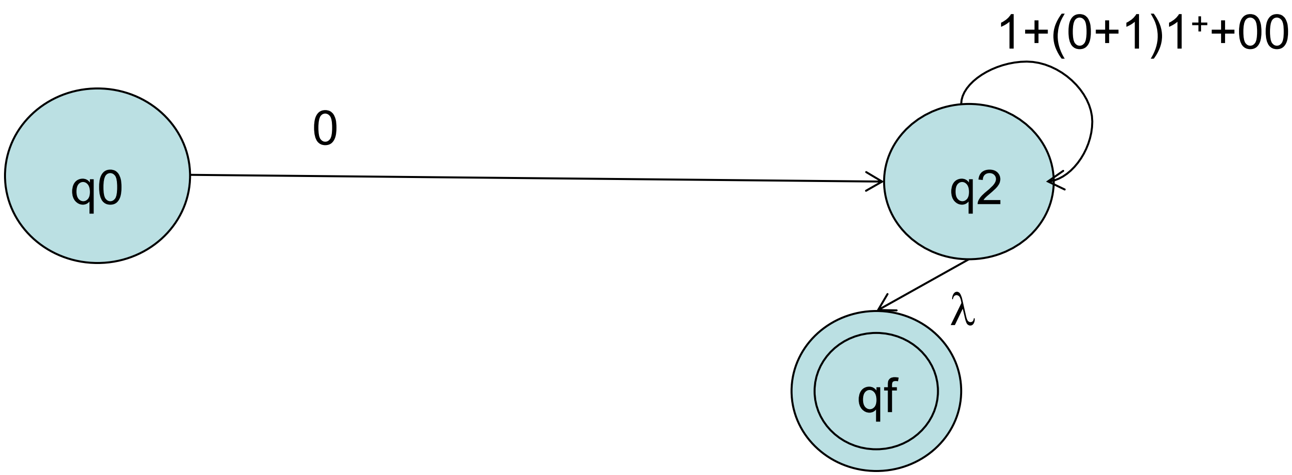


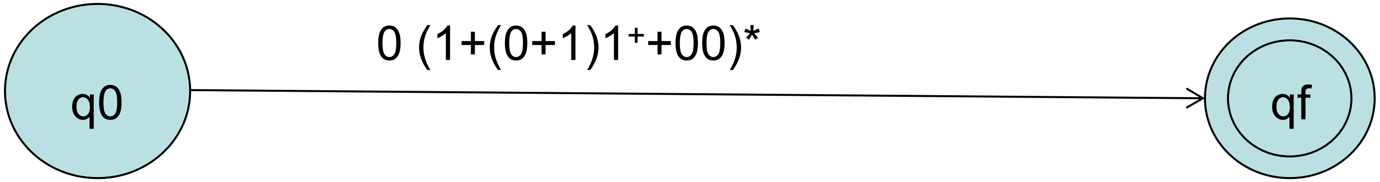
Use Ripping; Rip q1





Use Ripping; Rip q2





$L = 0\ (1+(0+1)1^{+}+00)^{*} = 0\ (1+(0+1)1^{+}+00)^{*}$

Regular Equations

- Assume that R, Q and P are sets such that P does not contain the string of length zero, and R is defined by
- $R = Q + RP$
- We wish to show that
- $R = QP^*$

Show QP* is a Solution

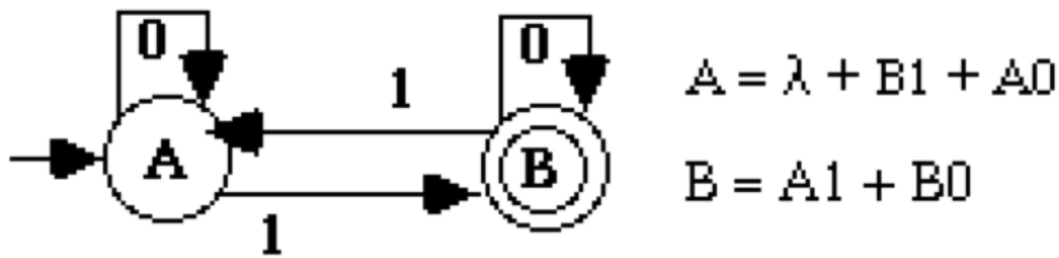
- We first show that QP* is contained in R. By definition, R = Q + RP.
- To see if QP* is a solution, we insert it as the value of R in Q + RP and see if the equation balances
- $R = Q + QP^*P = Q(\lambda+P^*P) = QP^*$
- Hence QP* is a solution, but not necessarily the only solution.

Uniqueness of Solution

- To prove uniqueness, we show that R is contained in QP*.
- By definition, $R = Q+RP = Q+(Q+RP)P$
- $= Q+QP+RP^2 = Q+QP+(Q+RP)P^2$
- $= Q+QP+QP^2+RP^3$
- ...
- $= Q(\lambda+P+P^2+ \ldots +P^i)+RP^{i+1}$, for all $i>=0$
- Choose any w in R, where $|W| = k$. Then, from above,
- $R = Q(\lambda+P+P^2+ \ldots +P^k)+RP^{k+1}$
- but, since P does not contain the string of length zero, w is not in RP^{k+1}. But then w is in
- $Q(\lambda+P+P^2+ \ldots +P^k)$ and hence w is in QP*.

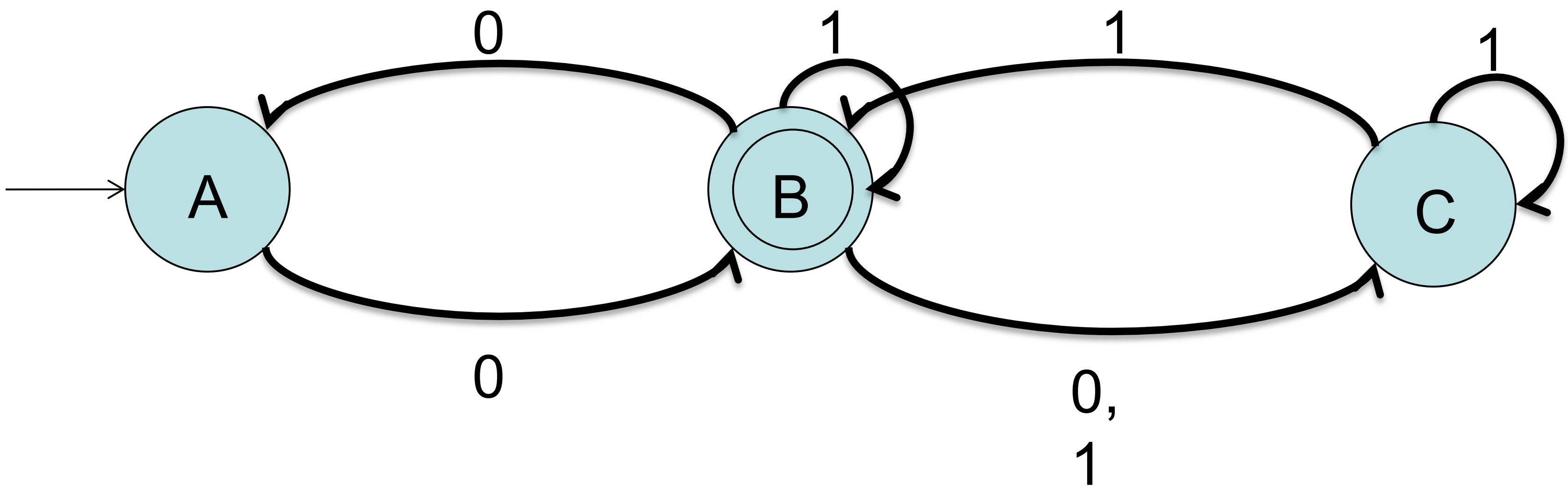
Example

- We use the above to solve simultaneous regular equations. For example, we can associate regular expressions with finite state automata as follows
- Hence,
- For A, Q=λ+B1; P=0
  A = QP* = (λ+B1)0*
  = B10* + 0*



- B = B10*1 + B0 + 0*1
  For B, Q=0*1; P= B10*1 + B0 = B(10*1 + 0)
- and therefore
- B = 0*1(10*1 + 0)*
- Note: This technique fails if there are lambda transitions.

Using Regular Equations



$A = \lambda + B0$

$B = A0 + C1 + B1$

$C = B(0+1) + C1$; $C = B(0+1)1^*$

$B = 0 + B00 + B(0+1)1^+ + B1$

$B = 0 + B\ (00+(0+1)\ 1^+ + 1)$; $B = 0(00 +(0+1)1^+ + 1)^*$

This is same form as with state ripping. It won't always be so.

State Minimization

- First step is to remove any state that is unreachable from the start state; a depth first search rooted at start state will identify all reachable states
- One seeks to merge compatible states – states q and s are compatible if, for all strings x, $\delta^*(q,x)$ and $\delta^*(s,x)$ are either both an accepting or both rejecting states
- One approach is to discover incompatible states – states q and s are incompatible if there exists a string x such that one of $\delta^*(q,x)$ and $\delta^*(s,x)$ is an accepting state and the other is not
- There are many ways to approach this, but my favorite is to do incompatible states via an n by n lower triangular matrix

Sample Minimization

- This uses a transition table
- Just an X denotes Immediately incompatible
- Pairs are dependencies for compatibility
- If a dependent is incompatible, so are pairs that depend on it
- When done, any not x--ed out are compatible
- Here, new states are <1,3>, <2,4,5>, <6>; <1,3> is start and not accept; others are accept
- Write new diagram

	a	b	c
>1	5	2	2
2	1	6	2
3	2	4	5
4	3	6	2
5	3	6	5
6	1	3	4

2	X				
3	2,5 2,4	X			
4	X	1.3	X		
5	X	1.3	X	2,5	
6	X	3,6 X 2,4	X	1,3 3,6 X 2,4	1,3 3,6 X 4,5
	1	2	3	4	5

Reversal of Regular Sets

- It is easier to do this with regular sets than with DFAs
- Let E be some arbitrary expression; E^R is formed by
  - Primitives: $\varnothing^R = \varnothing$ $\lambda^R = \lambda$ $a^R = a$
  - Closure:
    - $(A \cdot B)^R = (B^R \cdot A^R)$
    - $(A + B)^R = (A^R + B^R)$
    - $(A^*)^R = (A^{R*})$
- Challenge: How would you do this with FSA models?
  - Start with DFA; change all final to start states; change start to a final state; and reverse edges
  - Note that this creates multiple start states; can create a new start state with λ-transitions to multiple starts

Substitution

- A substitution is a function, f, from each member, a, of an alphabet, Σ, to a language L_a
- Regular languages are closed under substitution of regular languages (i.e., each L_a is regular)
- Easy to prove by replacing each member of Σ in a regular expression for a language L with regular expression for L_a
- A homomorphism is a substitution where each L_a is a single string

Quotient with Regular Sets

- Quotient of two languages B and C, denoted B/C, is defined as $B/C = \{x \mid \exists y \in C \text{ where } xy \in B\}$
- Let B be recognized by DFA
  $A_B = (Q_B,\Sigma,\delta_B,q_{1B},F_B)$ and C by
  $A_C = (Q_C,\Sigma,\delta_C,q_{1C},F_C)$
- Define the recognizer for B/C by
  $A_{B/C} = (Q_B \cup Q_B \times Q_C,\Sigma,\delta_{B/C},q_{1B}, F_B \times F_C)$
  $\delta_{B/C}(q,a) = \{\delta_B(q,a)\}$ $\quad a\in\Sigma, q\in Q_B$
  $\delta_{B/C}(q,\lambda) = \{<q,q_{1C}>\}$ $\quad q\in Q_B$
  $\delta_{B/C}(<q,p>,\lambda) = \{\delta_B(q,a),\delta_C(p,a)\}$ $\quad a\in\Sigma, q\in Q_B, p\in Q_C$
- The basic idea is that we simulate B and then randomly decide it has seen x and continue by looking for y, simulating B continuing after x but with C starting from scratch

Quotient Again

- Assume some class of languages, $\mathfrak{C}$, is closed under concatenation, intersection with regular and substitution of members of $\mathfrak{C}$, show $\mathfrak{C}$ is closed under Quotient with Regular
- $L/R = \{ x \mid \exists y \in R \text{ where } xy \in L \}$
  - Define $\Sigma' = \{ a' \mid a \in \Sigma \}$
  - Let $h(a) = a$; $h(a') = \lambda$ where $a \in \Sigma$
  - Let $g(a) = a'$ where $a \in \Sigma$
  - Let $f(a) = \{a, a'\}$ where $a \in \Sigma$
  - $L/R = h(f(L) \cap (\Sigma^* \cdot g(R)))$

Applying Meta Approach

- $\text{INIT}(L) = \{\, x \mid \exists y \in \Sigma^* \text{ where } xy \in L \,\}$
  - $\text{INIT}(L) = h(\, f(L) \cap (\, \Sigma^* \cdot g(\Sigma^*) \,)\,)$
  - Also $\text{INIT}(L) = L \,/\, \Sigma^*$
- $\text{LAST}(L) = \{\, y \mid \exists x \in \Sigma^* \text{ where } xy \in L \,\}$
  - $\text{LAST}(L) = h(\, f(L) \cap (\, g(\Sigma^*) \cdot \Sigma^* \,)\,)$
- $\text{MID}(L) = \{\, y \mid \exists x,z \in \Sigma^* \text{ where } xyz \in L \,\}$
  - $\text{MID}(L) = h(\, f(L) \cap (\, g(\Sigma^*) \cdot \Sigma^* \cdot g(\Sigma^*) \,)\,)$
- $\text{EXTERIOR}(L) = \{\, xz \mid \exists y \in \Sigma^* \text{ where } xyz \in L \,\}$
  - $\text{EXTERIOR}(L) = h(\, f(L) \cap (\, \Sigma^* \cdot g(\Sigma^*) \cdot \Sigma^* \,)\,)$

Making Life Easy

- The key in proving closure is to always try to identify the "best" equivalent formal model for regular sets when trying to prove a particular property
- For example, how could you even conceive of proving closure under intersection and complement in regular expression notations?
- Note how much easier quotient is when have closure under concatenation, and substitution and intersection with regular languages than showing in FSA notation

Reachable and Reaching

- Reachable*from*(q) = { p | $\exists$w $\ni$ δ(q,w)=p }
  - Just do depth first search from q, marking all reachable states. Works for NFA as well.
- Reaching*to*(q) = { p | $\exists$w $\ni$ δ(p,w)=q }
  - Do depth first from q, going backwards on transitions, marking all reaching states. Works for NFA as well.

Min and Max

- $\mathrm{Min}(L) = \{ w \mid w \in L$ and no proper prefix of w is in $L \} =$
  $\{ w \mid w \in L$ and if $w=xy$, $x \in \Sigma^*$, $y \in \Sigma^+$ then $x \notin L\}$
- $\mathrm{Max}(L) = \{ w \mid w \in L$ and w is not the proper prefix of any word in $L \} =$
  $\{ w \mid w \in L$ and if $y \in \Sigma^+$ then $wy \notin L \}$
- Examples:
  - $\mathrm{Min}(0(0+1)^*) = \{0\}$
  - $\mathrm{Max}(0(0+1)^*) = \{\}$
  - $\mathrm{Min}(01 + 0 + 10) = \{0,10\}$
  - $\mathrm{Max}(01 + 0 + 10) = \{01,10\}$
  - $\mathrm{Min}(\{a^ib^jc^k \mid i \le k \text{ or } j \le k\}) = \{a^ib^jc^k \mid \mid i,j \ge 0, k = \min(i, j)\}$
  - $\mathrm{Max}(\{a^ib^jc^k \mid i \le k \text{ or } j \le k\}) = \{\}$ because k has no bound
  - $\mathrm{Min}(\{a^ib^jc^k \mid i \ge k \text{ or } j \ge k\}) = \{\lambda\}$
  - $\mathrm{Max}(\{a^ib^jc^k \mid i \ge k \text{ or } j \ge k\}) = \{a^ib^jc^k \mid \mid i,j \ge 0, k = \max(i, j)\}$

Regular Closed under Min

- Assume L is regular then Min(L) is regular
- Let L= L(A), where A = (Q,Σ,δ,q_0,F) is a DFA with no state unreachable from q_0
- Define $A_{min} = (Q\cup\{dead\},\Sigma,\delta_{min},q_0,F)$, where for $a\in\Sigma$ $\delta_{min}(q,a) = \delta(q,a)$, if $q\in Q-F$; $\delta_{min}(q,a) = dead$, if $q\in F$; $\delta_{min}(dead,a) = dead$

The reasoning is that the machine A_{min} accepts only elements in L that are not extensions of shorter strings in L. By making it so transitions from all final states in A_{min} go to the new "dead" state, we guarantee that extensions of accepted strings will not be accepted by this new automaton.

Therefore, Regular Languages are closed under Min.

Regular Closed under Max

- Assume L is regular then Max(L) is regular
- Let L= *L*(A), where A = (Q,Σ,δ,q_0,F) is a DFA with no state unreachable from q_0
- Define $A_{max} = (Q,\Sigma,\delta,q_0,F_{max})$, where
  $F_{max} = \{ f \mid f \in F \text{ and } \text{Reachable}\mathit{from}^+(f) \cap F = \Phi \}$
  where $\text{Reachable}\mathit{from}^+(q) = \{ p \mid \exists w \ni |w|>0 \text{ and } \delta(q,w) = p \}$

The reasoning is that the machine A_{max} accepts only elements in L that cannot be extended. If there is a non-empty string that leads from some final state f to any final state, including f, then f cannot be final in A_{max}. All other final states can be retained.

The inductive definition of Reachable*from*$^+$ is:

1. Reachable*from*$^+$(q) contains $\{ s \mid$ there exists an element of Σ, a, such that $\delta(q,a) = s \}$
2. If s is in Reachable*from*$^+$ (q) then Reachable*from*$^+$ (q) contains $\{ t \mid$ there exists an element of Σ, a, such that $\delta(s,a) = t \}$
3. No other states are in Reachable*from*$^+$(q)

Therefore, Regular Languages are closed under Max.

Pumping Lemma Concept

- Let A = (Q,Σ,δ,q_1,F) be a DFA, where $Q = \{q_1,q_2, \dots , q_N\}$
- The "pigeon hole principle" tells us that whenever we visit N+1 or more states, we must visit at least one state more than once (loop)
- Any string, w, of length N or greater leads to us making N transitions after visiting the start state, and so we visit at least one state more than once when reading w

Pumping Lemma For Regular

- Theorem: Let L be regular then there exists an $N>0$ such that, if $w \in L$ and $|w| \geq N$, then w can be written in the form xyz, where $|xy| \leq N$, $|y|>0$, and for all $i \geq 0$, $xy^iz \in L$
- This means that interesting regular languages (infinite ones) have a very simple self-embedding property that occurs early in long strings

Pumping Lemma Proof

- If L is regular then it is recognized by some DFA, $A=(Q,\Sigma,\delta,q_0,F)$. Let $|Q| = N$ states. For any string w, such that $|w| \geq N$, A must make N+1 state visits to consume its first N characters, followed by $|w|-N$ more state visits.
- In its first N+1 state visits, A must enter at least one state two or more times.
- Let $w = v_1 \ldots v_j \ldots v_k \ldots v_m$, where $m = |w|$, and $\delta(q_0,v_1 \ldots v_j)=\delta(q_0,v_1 \ldots v_k)$, $k > j$, and let this state represent the first one repeated while A consumes w.
- Define $x = v_1 \ldots v_j$, $y = v_{i+1} \ldots v_k$, and $z = v_{k+1} \ldots v_m$. Clearly w=xyz. Moreover, since $k > j$, $|y| > 0$, and since $k \leq N$, $|xy| \leq N$.
- Since A is deterministic, $\delta(q_0,xy)=\delta(q_0,xy^i)$, for all $i \geq 0$.
- Thus, if $w \in L$, $\delta(q_0,xyz) \in F$, and so $\delta(q_0,xy^iz) \in F$, for all $i \geq 0$.
- Consequently, if $w \in L$, $|w| \geq N$, then w can be written in the form xyz, where $|xy| \leq N$, $|y| > 0$, and for all $i \geq 0$, $xy^iz \in L$.

Lemma's Adversarial Process

- Assume L = $\{a^n b^n \mid n>0\}$ is regular
- P.L.: Provides N > 0
  - We CANNOT choose N; that's the P.L.'s job
- Our turn: Choose $a^N b^N \in L$
  - We get to select a string in L
- P.L.: $a^N b^N = xyz$, where $|xy| \leq N$, $|y| > 0$, and for all $i \geq 0$, $xy^i z \in L$
  - We CANNOT choose split, but P.L. is constrained by N
- Our turn: Choose i = 0.
  - We have the power here
- P.L: $a^{N-|y|} b^N \in L$; just a consequence of P.L.
- Our turn: $a^{N-|y|} b^N \notin L$; just a consequence of L's structure
- CONTRADICTION, so L is <u>NOT</u> regular

xwx is not Regular (PL)

- **L = { x w x | x,w∈{a,b}+} :**
- Assume that L is Regular.
- PL: Let N > 0 be given by the Pumping Lemma.
- YOU: Let s be a string, s ∈ L, such that s = $a^N baa^N b$
- PL: Since s ∈ L and |s| ≥ N, s can be split into 3 pieces, s = xyz, such that |xy| ≤ N and |y| > 0 and ∀ i ≥ 0 xy^iz ∈ L
- YOU: Choose i = 2
- PL: $xy^2z = xyyz \in L$
- Thus, $a^{N + |y|}baa^Nb$ would be in L, but this is not so since N+|y| ≠ N
- We have arrived at a contradiction.
- Therefore L is not Regular.

$a^{Fib(k)}$ is not Regular (PL)

- **L = {$a^{Fib(k)}$ | k>0} :**
- Assume that L is regular
- Let N be the positive integer given by the Pumping Lemma
- Let *s* be a string **s = $a^{Fib(N+3)} \in$ L**
- Since *s* $\in$ L and |s| ≥ N (Fib(N+3)>N in all cases; actually Fib(N+2)>N as well), s is split by PL into xyz, where |xy| ≤ N and |y| > 0 and for all i ≥ 0, $xy^iz \in$ L
- We choose i = 2; by PL: xy^2z = xyyz$\in$ L
- Thus, $a^{Fib(N+3)+|y|}$ would be $\in$ L. This means that there is a Fibonacci number between Fib(N+3) and Fib(N+3)+N, but the smallest Fibonacci greater than Fib(N+3) is Fib(N+3)+Fib(N+2) and Fib(N+2)>N
  This is a contradiction, therefore L is not regular ■
- Note: Using values less than N+3 could be dangerous because N could be 1 and both Fib(2) and Fib(3) are within N (1) of Fib(1).

Myhill-Nerode Theorem

The following are equivalent:

1. L is accepted by some DFA
2. L is the union of some of the classes of a right invariant equivalence relation, R, of finite index.
3. The specific right invariance equivalence relation R_L where x R_L y iff $\forall$z [xz $\in$ L iff yz $\in$ L]
   has finite index

Definition. R is a right invariant equivalence relation iff R is an equivalence relation and $\forall$z [x R y implies xz R yz].

Note: This is only meaningful for relations over strings.

Myhill-Nerode 1 ⇒ 2

1. Assume L is accepted by some DFA, $A = (Q,\Sigma,\delta,q_1,F)$
2. Define R_A by $x \, R_A \, y$ iff $\delta^*(q_1,x) = \delta^*(q_1,y)$. First, R_A is defined by equality and so is obviously an equivalence relation (Clearly if $\delta^*(q_1,x) = \delta^*(q_1,y)$ then $\forall z \; \delta^*(q_1,xz) = \delta^*(q_1,yz)$ because A is deterministic. Moreover if $\forall z \; \delta^*(q_1,xz) = \delta^*(q_1,yz)$ then $\delta^*(q_1,x) = \delta^*(q_1,y)$, just by letting $z = \lambda$. Putting it together $x \, R_A \, y$ L iff $\forall z \; xz \, R_A \, yz$. Thus, R_A is right invariant; its index is $|Q|$ which is finite; and $L(A) = \cup_{\delta^*(x)\in F}[x]_{RA}$, where $[x]_{RA}$ refers to the equivalence class containing the string x.

Myhill-Nerode 2 $\Rightarrow$ 3

2. Assume L is the union of some of the classes of a right invariant equivalence relation, R, of finite index.
3. Since x R y iff $\forall$z [xz R yz], R is right invariant and L is the union of some of the equivalence classes, then
   $x \; R \; y \Rightarrow \forall z \, [\, xz \in L \text{ iff } yz \in L \,] \Rightarrow x \; R_L \; y$.
   This means that the index of R_L is less than or equal to that of R and so is finite. Note than the index of R_L is then less than or equal to that of any other right invariant equivalence relation, R, of finite index that defines L.

Myhill-Nerode 3 $\Rightarrow$ 1

3. Assume the specific right invariance equivalence relation R_L where $x \, R_L \, y$ iff $\forall z \, [\, xz \in L$ iff $yz \in L \,]$ has finite index
1. Define the automaton $A = (Q,\Sigma,\delta,q_1,F)$ by
   $Q = \{ \, [x]_{R_L} \mid x \in \Sigma^* \, \}$
   $\delta([x]_{R_L},a) = [xa]_{R_L}$
   $q1 = [\lambda]$
   $F = \{ \, [x]_{R_L} \mid x \in L \, \}$

   Note: This is the minimum state automaton and all others are either equivalent or have redundant indistinguishable states

Use of Myhill-Nerode

- $L = \{a^n b^n \mid n>0\}$ is NOT regular.
- Assume otherwise.
- M-N says that the specific r.i. equiv. relation R_L has finite index, where $x\ R_L\ y$ iff $\forall z\ [\ xz \in L$ iff $yz \in L\ ]$.
- Consider the equivalence classes $[a^i b]$ and $[a^j b]$, where $i,j>0$ and $i \neq j$.
- $a^i b b^{i-1} \in L$ but $a^j b b^{i-1} \notin L$ and so $[a^i b]$ is not related to $[a^j b]$ under R_L and thus $[a^i b] \neq [a^j b]$.
- This means that R_L has infinite index.
- Therefore L is not regular.

xwx is not Regular (MN)

- **L = { x a x | x∈{a,b}+} :**
- We consider the right invariant equivalence class $[a^ib]$, $i>0$.
- It's clear that a^ibaa^ib is in the language, but a^kbaa^ib is not when $k < i$.
- This shows that there is a separate equivalence class, $[a^ib]$, induced by R_L, for each $i>0$. Thus, the index of R_L is infinite and Myhill-Nerode states that L cannot be Regular.

$a^{Fib(k)}$ is not Regular (MN)

- **L = {$a^{Fib(k)}$ | k>0} :**
- We consider the collection of right invariant equivalence classes [$a^{Fib(j)}$], j > 2.
- It's clear that $a^{Fib(j)}a^{Fib(j+1)}$ is in the language, but $a^{Fib(k)}a^{Fib(j+1)}$ is not when k>2 and k≠j and k≠j+2
- This shows that there is a separate equivalence class [$a^{Fib(j)}$] induced by R_L, for each j > 2.
- Thus, the index of R_L is infinite and Myhill-Nerode states that L cannot be Regular.

$a^n b^m$ m≠m is not Regular (MN)

- **L = {$a^n b^m$ | n≠m} :**
- We consider the collection of right invariant equivalence classes $[a^i]$, $i \geq 0$.
- It's clear that $a^i b^i$ is not in L, but $a^j b^i$ is when $j \neq i$
- This shows that there is a separate equivalence class $[a^i]$ induced by R_L, for each $i \geq 0$.
- Thus, the index of R_L is infinite and Myhill-Nerode states that L cannot be Regular.

Myhill-Nerode and Minimization

- Corollary: The minimum state DFA for a regular language, L, is formed from the specific right invariance equivalence relation R_L where
  $x \; R_L \; y$ iff $\forall z$ [$xz \in L$ iff $yz \in L$]
- Moreover, all minimum state machines have the same structure as the above, except perhaps for the names of states

What is Regular So Far?

- Any language accepted by a DFA
- Any language accepted by an NFA
- Any language specified by a Regular Expression
- Any language representing the unique solution to a set of properly constrained regular equations

What is <u>NOT</u> Regular?

- Well, anything for which you cannot write an accepting DFA or NFA, or a defining regular expression, or a right/left linear grammar, or a set of regular equations, but that’s not a very useful statement
- There are two tools we have:
  – Pumping Lemma for Regular Languages
  – Myhill-Nerode Theorem

Finite State Transducers

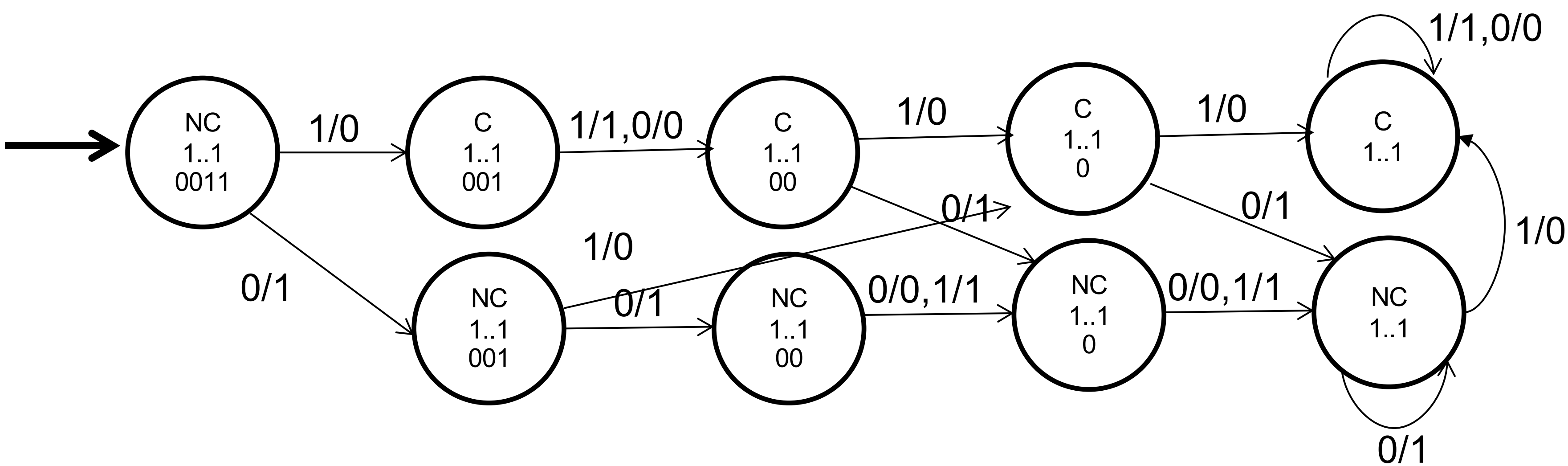
- A transducer is a machine with output
- Mealy Model
  - M = $(Q, \Sigma, \Gamma, \delta, \gamma, q_0)$

    Γ is the finite output alphabet

    γ: $Q \times \Sigma \rightarrow \Gamma$ is the output function
  - Essentially a Mealy Model machine produced a character of output for each character of input it consumes, and it does so on the transitions from one state to the next.
  - A Mealy Model represents a synchronous circuit whose output is triggered each time a new input arrives.

Sample Mealy Model

- Write a Mealy finite state machine that produces the 2's complement result of subtracting 1101 from a binary input stream (assuming at least 4 bits of input)



Finite State Transducers

- Moore Model
  - $M = (Q, \Sigma, \Gamma, \delta, \gamma, q_0)$

    Γ is the finite output alphabet

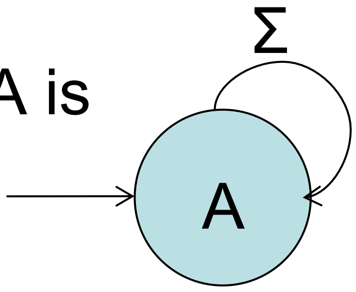
    $\gamma: Q \rightarrow \Gamma$ is the output function
  - Essentially a Moore Model machine produced a character of output whenever it enters a state, independent of how it arrived at that state.
  - A Moore Model represents an asynchronous circuit whose output is a steady state until new input arrives.

Decision and Closure Properties

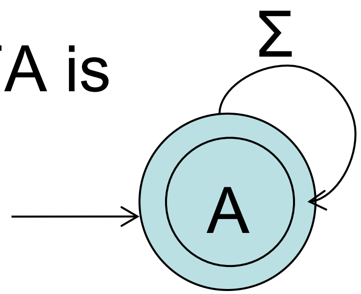
Regular Languages

Decidable Properties

- Membership (just run DFA over string)
- L = Ø: Minimize and see if minimum state DFA is



- L = Σ*: Minimize and see if minimum state DFA is



- Finiteness: Minimize and see if there are no loops emanating from a final state
- Equivalence: Minimize both and see if isomorphic

Closure Properties

- Virtually everything with members of its own class as we have already shown

- Union, concatenation, Kleene *, complement, intersection, set difference, reversal, substitution, homomorphism, quotient with regular sets, Prefix, Suffix, Substring, Exterior, Min, Max and so much more

Formal Languages

History of Formal Language

- In 1940s, Emil Post (mathematician) devised rewriting systems as a way to describe how mathematicians do proofs. Purpose was to mechanize them.
- Early 1950s, Noam Chomsky (linguist) developed a hierarchy of rewriting systems (grammars) to describe natural languages.
- Late 1950s, Backus-Naur (computer scientists) devised BNF (a variant of Chomsky's context-free grammars) to describe the programming language Algol.
- 1960s was the time of many advances in parsing. In particular, parsing of context free was shown to be no worse than $O(n^3)$. More importantly, useful subsets were found that could be parsed in $O(n)$.

Formalism for Grammars

Definition : A language is a set of strings of characters from some alphabet.

The strings of the language are called sentences or statements.

A string over some alphabet is a finite sequence of symbols drawn from that alphabet.

A meta-language is a language that is used to describe another language.

A very well known meta-language is BNF (Backus Naur Form)

It was developed by John Backus and Peter Naur, in the late 50s, to describe programming languages.

Noam Chomsky in the early 50s developed context free grammars that can be expressed using BNF.

Grammars

- $G = (V, \Sigma, R, S)$ is a Phrase Structured Grammar (PSG) where
  - V: Finite set of non-terminal symbols
  - Σ: Finite set of terminal symbols
  - R: finite set of rules of form $\alpha \to \beta$,
    - α in $(V \cup \Sigma)^* V (V \cup \Sigma)^*$
    - β in $(V \cup \Sigma)^*$
  - S: a member of V called the start symbol
- Right linear restricts all rules to be of forms
  - α in V
  - β of form ΣV, Σ or λ

Derivations

- $x \Rightarrow y$ reads as x derives y iff
  - $x = \gamma\alpha\delta$, $y = \gamma\beta\delta$ and $\alpha \rightarrow \beta$
- $\Rightarrow^*$ is the reflexive, transitive closure of $\Rightarrow$
- $\Rightarrow^+$ is the transitive closure of $\Rightarrow$
- $x \Rightarrow^* y$ iff $x = y$ or $x \Rightarrow^* z$ and $z \Rightarrow y$
- Or, $x \Rightarrow^* y$ iff $x = y$ or $x \Rightarrow z$ and $z \Rightarrow^* y$
- $L(G) = \{ w \mid S \Rightarrow^* w \}$ is the language generated by G.

Regular Grammars

- Regular grammars are also called right linear grammars
- Each rule of a regular grammar is constrained to be of one of the three forms:

  $A \rightarrow a$, $\quad A \in V$, $a \in \Sigma^*$

  $A \rightarrow \lambda$, $\quad A \in V$, $a \in \Sigma^*$

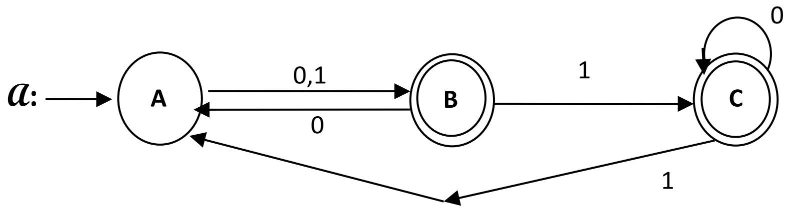
  $A \rightarrow aB$, $\quad A, B \in V$, $a \in \Sigma^*$

DFA to Regular Grammar

- Every language recognized by a DFA is generated by an equivalent regular grammar
- Given $A = (Q,\Sigma,\delta,q_0,F)$, $L(A)$ is generated by $G_A = (Q,\Sigma,R,q_0)$ where R contains
  $q \to as$ iff $\delta(q,a) = s$
  $q \to \lambda$ iff $q \in F$

Example of DFA to Grammar

- **DFA**



- **Grammar**

A	$\rightarrow$	**0 B**	**|**	**1 B**		
B	$\rightarrow$	**0 A**	**|**	**1 C**	**|**	λ
C	$\rightarrow$	**0 C**	**|**	**1 A**	**|**	λ

Regular Grammar to NFA

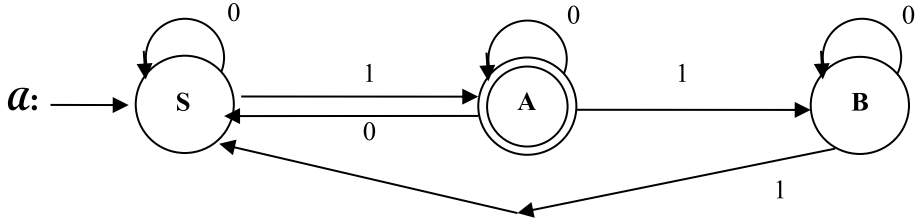
- Every language generated by a regular grammar is recognized by an equivalent NFA
- Given G = (V, Σ, R, S), L(G) is recognized by $A_G = (V \cup \{f\}, \Sigma, \delta, S, \{f\})$ where δ is defined by
  $\delta(A,a) \subseteq \{B\}$ iff $A \rightarrow aB$
  $\delta(A,a) \subseteq \{f\}$ iff $A \rightarrow a$
  $\delta(A,\lambda) \subseteq \{f\}$ iff $A \rightarrow \lambda$

Example of Grammar to NFA

- **Grammar**

S	$\rightarrow$	**0 S**	|	**1 A**			
A	$\rightarrow$	**0 S**	|	**0 A**	|	**1 B**	| λ
B	$\rightarrow$	**1 S**	|	**0 B**			

- **DFA**



What More is Regular?

- Any language, L, generated by a right linear grammar
- Any language, L, generated by a left linear grammar ($A \rightarrow a$, $A \rightarrow \lambda$, $A \rightarrow Ba$)
  - Easy to see L is regular as we can reverse these rules and get a right linear grammar that generates L^R, but then L is the reverse of a regular language which is regular
  - Similarly, the reverse L^R of any regular language L is right linear and hence the language itself is left linear
- Any language, L, that is the union of some of the classes of a right invariant equivalence relation of finite index

Mixing Right and Left Linear

- We can get non-Regular languages if we present grammars that have both right and left linear rules
- To see this, consider G = ({S,T}, Σ, R, S), where R is:
  - $S \rightarrow aT$
  - $T \rightarrow Sb \mid b$
- *L*(G) = { $a^n b^n \mid n > 0$ } which is a classic non-regular, context-free language

Context Free Languages

Context Free Grammar

$G = (V, \Sigma, R, S)$ is a PSG where

Each member of R is of the form

$A \rightarrow \alpha$ where α is a strings $(V \cup \Sigma)^*$

Note that the left hand side of a rule is a letter in V;

The right hand side is a string from the combined alphabets

The right hand side can even be empty (ε or λ)

A context free grammar is denoted as a CFG and the language generated is a Context Free Language (CFL).

A CFL is recognized by a Push Down Automaton (PDA) to be discussed a bit later.

Sample CFG

Example of a grammar for a small language:

G = ({<program>, <stmt-list>, <stmt>, <expression>},
{begin, end, ident, ;, =, +, -}, R, <program>) where R is

<program> → begin <stmt-list> end

<stmt-list> → <stmt> | <stmt> ; <stmt-list>

<stmt> → ident = <expression>

<expression> → ident + ident | ident - ident | ident

Here "ident" is a token return from a scanner, as are "begin", "end", ";", "=", "+", "-"

Note that ";" is a separator (Pascal style) not a terminator (C style).

Derivation

A sentence generation is called a derivation.

Grammar for a simple assignment statement:

R1 <assgn> → <id> := <expr>
R2 <id> → a | b | c
R3 <expr> → <id> + <expr>
R4 | <id> * <expr>
R5 | (<expr>)
R6 | <id>

In a **leftmost derivation** only the leftmost non-terminal is replaced

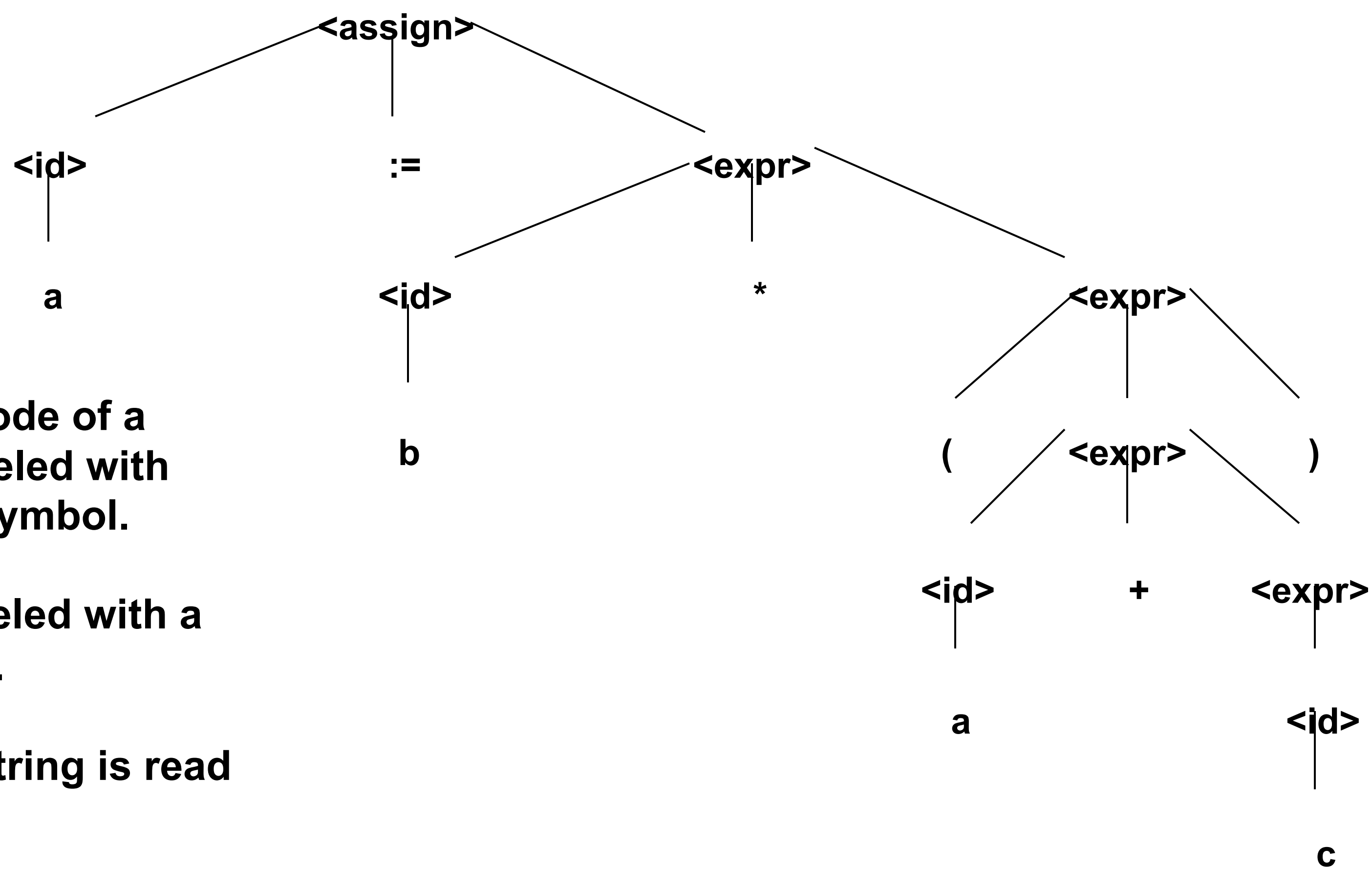
The statement a := b * (a + c)
Is generated by the leftmost derivation:

<assgn>	⇒ <id> := <expr>	R1
	⇒ a := <expr>	R2
	⇒ a := <id> * <expr>	R4
	⇒ a := b * <expr>	R2
	⇒ a := b * (<expr>)	R5
	⇒ a := b * (<id> + <expr>)	R3
	⇒ a := b * (a + <expr>)	R2
	⇒ a := b * (a + <id>)	R6
	⇒ a := b * (a + c)	R2

Parse Trees

<u>A parse tree is a graphical representation of a derivation</u>

For instance the parse tree for the statement a := b * (a + c) is:



Every internal node of a parse tree is labeled with a non-terminal symbol.

Every leaf is labeled with a terminal symbol.

The generated string is read left to right

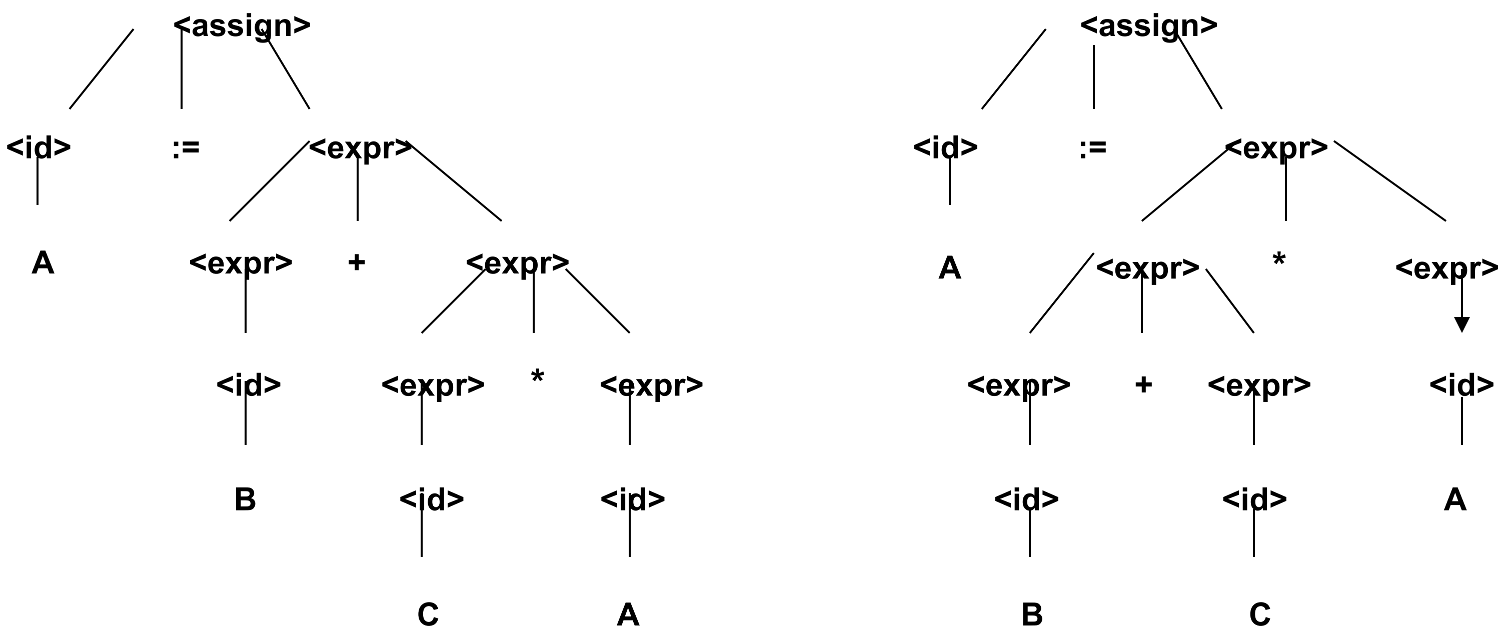
Ambiguity

A grammar that generates a sentence for which there are two or more distinct parse trees is said to be "ambiguous"

For instance, the following grammar is ambiguous because it generates distinct parse trees for the expression a := b + c * a

<assgn> → <id> := <expr>
<id> → a | b | c
<expr> → <expr> + <expr>
| <expr> * <expr>
| (<expr>)
| <id>

Ambiguous Parse



This grammar generates two parse trees for the same expression.

If a language structure has more than one parse tree, the meaning of the structure cannot be determined uniquely.

Precedence

Operator precedence:

If an operator is generated lower in the parse tree, it indicates that the operator has precedence over the operator generated higher up in the tree.

An unambiguous grammar for expressions:

<assign> → <id> := <expr>
<id> → a | b | c
<expr> → <expr> + <term>
| <term>
<term> → <term> * <factor>
| <factor>
<factor> → (<expr>)
| <id>

This grammar indicates the usual precedence order of multiplication and addition operators.

This grammar generates unique parse trees independently of doing a rightmost or leftmost derivation

Left (right)most Derivations

Leftmost derivation:

<assgn> → <id> := <expr>
→ a := <expr>
→ a := <expr> + <term>
→ a := <term> + <term>
→ a := <factor> + <term>
→ a := <id> + <term>
→ a := b + <term>
→ a := b + <term> *<factor>
→ a := b + <factor> * <factor>
→ a := b + <id> * <factor>
→ a := b + c * <factor>
→ a := b + c * <id>
→ a := b + c * a

Rightmost derivation:

<assgn> ⇒ <id> := <expr>
⇒ <id> := <expr> + <term>
⇒ <id> := <expr> + <term> *<factor>
⇒ <id> := <expr> + <term> *<id>
⇒ <id> := <expr> + <term> * a
⇒ <id> := <expr> + <factor> * a
⇒ <id> := <expr> + <id> * a
⇒ <id> := <expr> + c * a
⇒ <id> := <term> + c * a
⇒ <id> := <factor> + c * a
⇒ <id> := <id> + c * a
⇒ <id> := b + c * a
⇒ a := b + c * a

Ambiguity Test

- A Grammar is Ambiguous if there are two distinct parse trees for some string
- Or, two distinct leftmost derivations
- Or, two distinct rightmost derivations
- Some languages are inherently ambiguous but many are not
- Unfortunately (to be shown later) there is no systematic test for ambiguity of context free grammars

Unambiguous Grammar

When we encounter ambiguity, we try to rewrite the grammar to avoid ambiguity.

The ambiguous expression grammar:

<expr> → <expr> <op> <expr> | id | int | (<expr>)
<op> → + | - | * | /

Can be rewritten as:

<expr> → <term> | <expr> + <term> | <expr> - <term>
<term> → <factor> | <term> * <factor> | <term> / <factor>.
<factor> → id | int | (<expr>)

Parsing Problem

The parsing Problem: Take a string of symbols in a language (tokens) and a grammar for that language to construct the parse tree or report that the sentence is syntactically incorrect.

For correct strings:

Sentence + grammar → parse tree

For a compiler, a sentence is a program:

Program + grammar → parse tree

Types of parsers:

Top-down aka predictive (recursive descent parsing)

Bottom-up aka shift-reduce

Removing Left Recursion if doing Top Down

Given left recursive and non left recursive rules

$A \to A\alpha_1 \mid \ldots \mid A\alpha_n \mid \beta_1 \mid \ldots \mid \beta_m$

Can view as

$A \to (\beta_1 \mid \ldots \mid \beta_m)\ (\alpha_1 \mid \ldots \mid \alpha_n\ )^*$

Star notation is an extension to normal notation with obvious meaning

Now, it should be clear this can be done right recursive as

$A \to \beta_1 B \mid \ldots \mid \beta_m\ B$

$B \to \alpha_1 B \mid \ldots \mid \alpha_n B \mid \lambda$

Right Recursive Expressions

Grammar: Expr → Expr + Term | Term
Term → Term * Factor | Factor
Factor → (Expr) | Int

Fix: Expr → Term ExprRest
ExprRest → + Term ExprRest | λ
Term → Factor TermRest
TermRest → * Factor TermRest | λ
Factor → (Expr) | Int

Bottom Up vs Top Down

- Bottom-Up: Two stack operations
  - Shift (move input symbol to stack)
  - Reduce (replace top of stack α with A, when $A \rightarrow \alpha$)
  - Challenge is when to do shift or reduce and what reduce to do.
    - Can have both kinds of conflict
- Top-Down:
  - If top of stack is terminal
    - If same as input, read and pop
    - If not, we have an error
  - If top of stack is a non-terminal A
    - Replace A with some α, when $A \rightarrow \alpha$
    - Challenge is what A-rule to use

Chomsky Normal Form

- Each rule of a CFG is constrained to be of one of the three forms:
  $A \rightarrow a, \qquad A \in V, a \in \Sigma$
  $A \rightarrow BC, \qquad A,B,C \in V$
- If the language contains λ then we allow
  $S \rightarrow \lambda$
  and constrain all non-terminating rules of form to be
  $A \rightarrow BC, \qquad A \in V, B,C \in V\text{-}\{S\}$

Nullable Symbols

- Let $G = (V, \Sigma, R, S)$ be an arbitrary CFG
- Compute the set $\text{Nullable}(G) = \{A \mid A \Rightarrow^* \lambda\}$
- Nullable(G) is computed as follows
  $\text{Nullable}(G) \supseteq \{ A \mid A \rightarrow \lambda \}$
  Repeat
  $\quad \text{Nullable}(G) \supseteq \{ B \mid B \rightarrow \alpha \text{ and } \alpha \in \text{Nullable}^* \}$
  until no new symbols are added

Removal of λ-Rules

- Let G = (V, Σ, R, S) be an arbitrary CFG
- Compute the set Nullable(G)
- Remove all λ-rules
- For each rule of form $B \to \alpha A \beta$ where A is nullable, add in the rule $B \to \alpha\beta$
- The above has the potential to greatly increase the number of rules and add unit rules (those of form $B \to C$, where $B,C \in V$)
- If S is nullable, add new start symbol S_0, as new start state, plus rules $S_0, \to \lambda$ and $S_0 \to \alpha$, where $S \to \alpha$

Chains (Unit Rules)

- Let $G = (V, \Sigma, R, S)$ be an arbitrary CFG that has had its λ-rules removed
- For $A \in V$, $\text{Chain}(A) = \{ B \mid A \Rightarrow^* B, B \in V \}$
- Chain(A) is computed as follows
  $\text{Chain}(A) \supseteq \{ A \}$
  Repeat
  $\quad \text{Chain}(A) \supseteq \{ C \mid B \rightarrow C \text{ and } B \in \text{Chain}(A) \}$
  until no new symbols are added

Removal of Unit-Rules

- Let G = (V, Σ, R, S) be an arbitrary CFG that has had its λ-rules removed, except perhaps from start symbol
- Compute Chain(A) for all A$\in$V
- Create the new grammar G = (V, Σ, R, S) where R is defined by including for each A$\in$V, all rules of the form $A \rightarrow \alpha$, where $B \rightarrow \alpha \in R$, $\alpha \notin V$ and $B \in$ Chain(A)
  Note: A$\in$Chain(A) so all its non unit-rules are included

Non-Productive Symbols

- Let $G = (V, \Sigma, R, S)$ be an arbitrary CFG that has had its λ-rules and unit-rules removed
- Non-productive non-terminal symbols never lead to a terminal string (not productive)
- Productive(G) is computed by
  Productive(G) $\supseteq \{ A \mid A \rightarrow \alpha, \alpha \in \Sigma^* \}$
  Repeat
  Productive(G) $\supseteq \{ B \mid B \rightarrow \alpha, \alpha \in (\Sigma \cup \text{Productive})^* \}$
  until no new symbols are added
- Keep only those rules that involve productive symbols
- If no rules remain, grammar generates nothing

Unreachable Symbols

- Let $G = (V, \Sigma, R, S)$ be an arbitrary CFG that has had its λ-rules, unit-rules and non-productive symbols removed
- Unreachable symbols are ones that are inaccessible from start symbol
- We compute the complement (Useful)
- Useful(G) is computed by
  Useful(G) $\supseteq$ { S }
  Repeat
  Useful(G) $\supseteq$ { $C \mid B \rightarrow \alpha C \beta$, $C \in V \cup \Sigma$, $B \in$ Useful(G) }
  until no new symbols are added
- Keep only those rules that involve useful symbols
- If no rules remain, grammar generates nothing

Reduced CFG

- A reduced CFG is one without λ-rules (except possibly for start symbol), no unit-rules, no non-productive symbols and no useless symbols

CFG to CNF

- Let $G = (V, \Sigma, R, S)$ be arbitrary reduced CFG
- Define $G' = (V \cup \{<a> | a \in \Sigma\}, \Sigma, R, S)$
- Add the rules $<a> \to a$, for all $a \in \Sigma$
- For any rule, $A \to \alpha$, $|\alpha| > 1$, change each terminal symbol, a, in α to the non-terminal $<a>$
- Now, for each rule $A \to BC\alpha$, $|\alpha| > 0$, introduce the new non-terminal $B<C\alpha>$, and replace the rule $A \to BC\alpha$ with the rule $A \to B<C\alpha>$ and add the rule $<C\alpha> \to C\alpha$
- Iteratively apply the above step until all rules are in CNF

Example of CNF Conversion

Starting Grammars

- $L = \{ a^i b^j c^k \mid i=j \text{ or } j=k \}$
- $G = (\{S,A,<B=C>,C,<A=B>\}, \{a,b\}, R, S)$
- R:
  - $S \rightarrow A \mid C$
  - $A \rightarrow a\ A \mid <B=C>$
  - $<B=C> \rightarrow b\ <B=C>\ c \mid \lambda$
  - $C \rightarrow C\ c \mid <A=B>$
  - $<A=B> \rightarrow a\ <A=B>\ b \mid \lambda$

Remove Null Rules

- **Nullable = {<B=C>, <A=B>, A, C, S}**
  - **S’ → S | λ** **// S’ added to V**
  - **S → A | C**
  - **A → a A | a |<B=C>**
  - **<B=C> → b <B=C> c | b c**
  - **C → C c | c | <A=B>**
  - **<A=B> → a <A=B> b | ab**

Remove Unit Rules

- **Chains= {[S':S',S,A,C,<A=B>,<B=C>],[S:S,A,C,<A=B>,<B=C>], [A:A,<B=C>],[C:C,<B=C>],[<B=C>:<B=C>], [<A=B>:<A=B>]}**
  - **S' → λ | aA | a | b<B=C>c | bc | Cc | c | a<A=B>b | ab**
  - **S → aA | a | b<B=C>c | bc | Cc | c | a<A=B>b | ab**
  - **A → aA | a | b<B=C>c | bc**
  - **<B=C> → b<B=C>c | bc**
  - **C → Cc | c | a<A=B>b | ab**
  - **<A=B> → a<A=B>b | ab**

Remove Useless Symbols

- All non-terminal symbols are productive (lead to terminal string)

- S is useless as it is unreachable from S’ (new start).
- All other symbols are reachable from S’

Normalize rhs as CNF

- S’ → λ | <a>A | a | <b><<B=C><c>> | <b><c> | C<c> | c | <a><<A=B><b>> | <a><b>
- A → <a>A | a |<b><<B=C><c>> | <b><c>
- <B=C> → <b><<B=C><c>> | <b><c>
- C → C<c> | c | <a><<A=B><b>> | <a><b>
- <A=B> → <a> <<A=B><b>> | <a><b>
- <<B=C><c>> → <B=C><c>
- <<A=B><b>> → <A=B><b>
- <a> → a
- <b> → b
- <c> → c

CKY (Cocke, Kasami, Younger) $O(N^3)$ PARSING

Dynamic Programming

To solve a given problem, we solve small parts of the problem (subproblems), then combine the solutions of the subproblems to reach an overall solution.

The Parsing problem for arbitrary CFGs was elusive, in that its complexity was unknown until the late 1960s. In the meantime, theoreticians developed notion of simplified forms that were as powerful as arbitrary CFGs. The one most relevant here is the Chomsky Normal Form – CNF. It states that the only rule forms needed are:

$A \rightarrow$	BC	where B and C are non-terminals
$A \rightarrow$	a	where a is a terminal

This is provided the string of length zero is not part of the language.

CKY (Bottom-Up Technique)

Let the input string be a sequence of n letters a_1 ... a_n.

Let the grammar contain r terminal and nonterminal symbols R_1 ... R_r,

Let R_1 be the start symbol.

Let P[n,n,r] be an array of Booleans. Initialize all elements of P to false.

For each i = 1 to n

  For each unit production $R_j \to a_i$, set P[i,1,j] = true.

  For each i = 2 to n

    For each j = 1 to n-i+1

      For each k = 1 to i-1

        For each production R_A -> R_B R_C

          If P[j,k,B] and P[j+k,i-k,C] then set P[j,i,A] = true

If P[1,n,1] is true then a_1 ... a_n is member of language

else a_1 ... a_n is not member of language

CKY Parser

Present the **CKY** recognition matrix for the string **abba** assuming the Chomsky Normal Form grammar, **G = ({S,A,B,C,D,E}, {a,b}, R, S)**, specified by the rules **R**:

S → AB | BA
A → CD | a
B → CE | b
C → a | b
D → AC
E → BC

	a	b	b	a
1	A,C	B,C	B,C	A,C
2	S,D	E	S,E	
3	B	B		
4	S,E			

2^{nd} CKY Example

E → E F | M E | P E | a
F → M F | P F | M E | P E
P → +
M → –

	a	–	a	+	a	–	a
1	E	M	E	P	E	M	E
2		E, F		E, F		E, F	
3	E		E		E		
4		E, F		E, F			
5	E		E				
6		E, F					
7	E						

CFL Pumping Lemma Concept

- Let L be a context free language the there is CNF grammar
  G = (V, Σ, R, S) such that *L*(G) = L.
- As G is in CNF all its rules that allow the string to grow are of the form
  A → BC, and thus growth has a binary nature.
- Any sufficiently long string z in L will have a parse tree that must have deep branches to accommodate z's growth.
- Because of the binary nature of growth, the width of a tree with maximum branch length k at its deepest nodes is at most 2^k; moreover, if the frontier of the tree is all terminal, then the string so produced is of length at most 2^{k-1}; since the last rule applied for each leaf is of the form A → a.
- Any terminal branch in a derivation tree of height > |V| has more than |V| internal nodes labelled with non-terminals. The "pigeon hole principle" tells us that whenever we visit |V| +1 or more nodes, we must use at least one variable label more than once. This creates a self-embedding property that is key to the repetition patterns that occur in the derivation of sufficiently long strings.

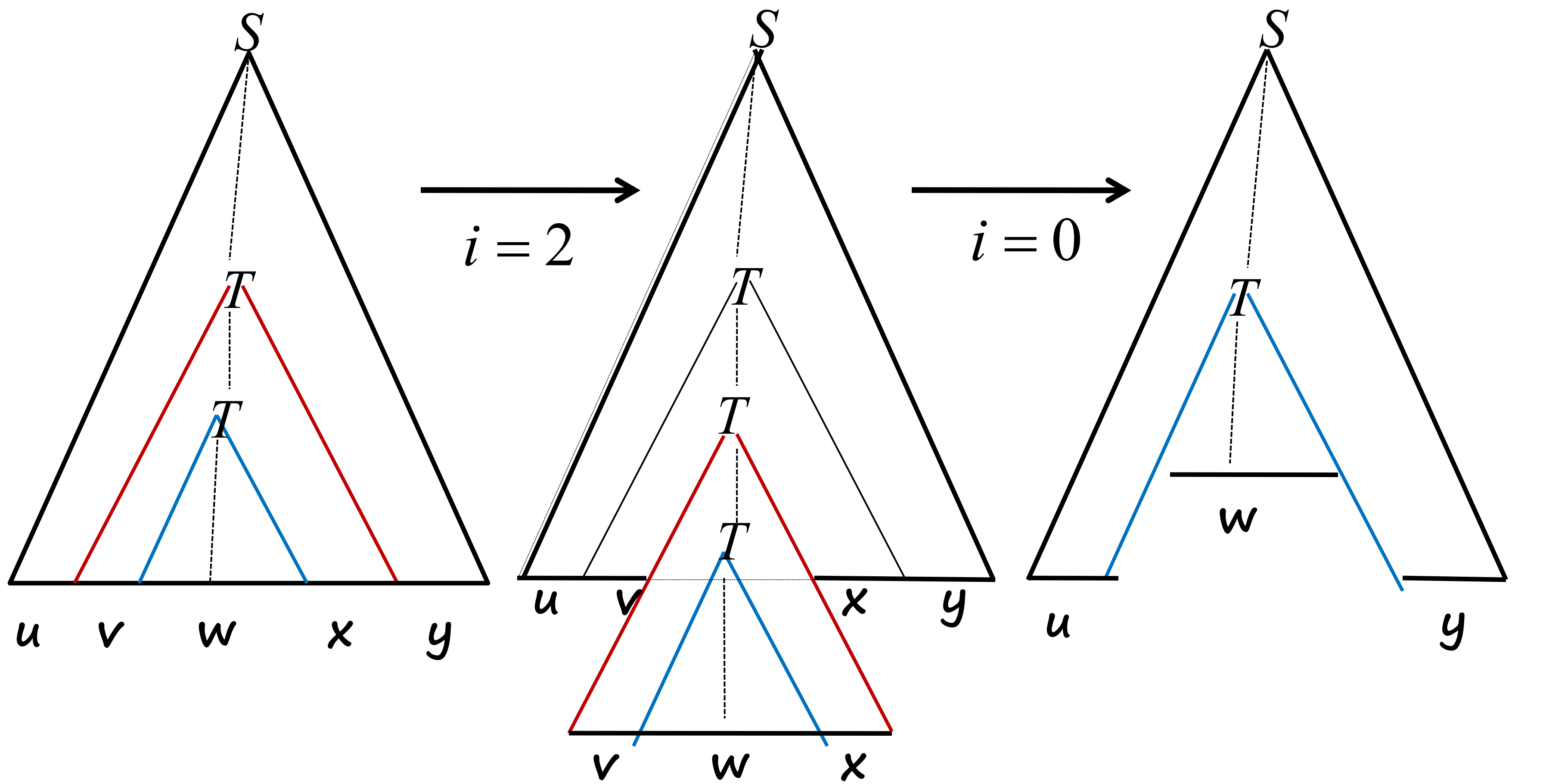
Pumping Lemma For CFL

- Let L be a CFL then there exists an $N>0$ such that, if $z \in L$ and $|z| \geq N$, then z can be written in the form uvwxy, where $|vwy| \leq N$, $|vx|>0$, and for all $i\geq 0$, $uv^iwx^iy \in L$.
- This means that interesting context free languages (infinite ones) have a self-embedding property that is symmetric around some central area, unlike regular where the repetition has no symmetry and occurs at the start.

Pumping Lemma Proof

- If L is a CFL then it is generated by some CNF grammar, $G = (V, \Sigma, R, S)$. Let $|V| = k$. For any string z, such that $|z| \geq N = 2^k$, the derivation tree for z based on G must have a branch with at least k+1 nodes labelled with variables from G.
- By the Pigeon Hole Principle at least two of these labels must be the same. Let the first repeated variable be T and consider the last two instances of T on this path.
- Let $z = uvwxy$, where $S \Rightarrow^* uTy \Rightarrow^* uvTxy \Rightarrow^* uvwxy$
- Clearly, then, we know $S \Rightarrow^* uTy$; $T \Rightarrow^* vTx$; and $T \Rightarrow^* w$
- But then, we can start with $S \Rightarrow^* uTy$; repeat $T \Rightarrow^* vTx$ zero or more times; and then apply $T \Rightarrow^* w$.
- But then, $S \Rightarrow^* uv^iwx^iy$ for all $i \geq 0$, and thus $uv^iwx^iy \in L$, for all $i \geq 0$.

Visual Support of Proof



Lemma's Adversarial Process

- Assume L = $\{a^n b^n c^n \mid n>0\}$ is a CFL
- P.L.: Provides N>0 We CANNOT choose N; that's the P.L.'s job
- Our turn: Choose $a^N b^N c^N \in L$ We get to select a string in L
- P.L.: $a^N b^N c^N = uvwxy$, where $|vwx| \leq N$, $|vx|>0$, and for all $i \geq 0$, $uv^i wx^i y \in L$ We CANNOT choose split, but P.L. is constrained by N
- Our turn: Choose i=0. We have the power here
- P.L: Two cases:
  (1) vwx contains some a's and maybe some b's. Because $|vwx| \leq N$, it cannot contain c's if it has a's. i=0 erases some a's but we still have N c's so $uwy \notin L$
  (2) vwx contains no a's. Because $|vx|>0$, vx contains some b's or c's or some of each. i=0 erases some b's and/or c's but we still have N a's so $uwy \notin L$
- CONTRADICTION, so L is <u>NOT</u> a CFL

Non-Closure

- Intersection ($\{ a^n b^n c^n \mid n \ge 0 \}$ is not a CFL)
  $\{ a^n b^n c^n \mid n \ge 0 \} =$
  $\{ a^n b^n c^m \mid n,m \ge 0 \} \cap \{ a^m b^n c^n \mid n,m \ge 0 \}$
  Both of the above are CFLs
- Complement
  If closed under complement then would be closed under Intersection as
  $A \cap B = \sim(\sim A \cup \sim B)$

Max and Min of CFL

- Consider the two operations on languages max and min, where
  – $\max(L) = \{ x \mid x \in L$ and, for no non-null y does $xy \in L \}$ and
  – $\min(L) = \{ x \mid x \in L$ and, for no proper prefix of x, y, does $y \in L \}$
- Describe the languages produced by max and min. for each of :
  – $L1 = \{ a^i b^j c^k \mid k \le i \text{ or } k \le j \}$ CFL
    - $\max(L1) = \{ a^i b^j c^k \mid k = \max(i, j) \}$ Non-CFL
    - $\min(L1) = \{ \lambda \}$ (string of length 0) Regular
  – $L2 = \{ a^i b^j c^k \mid k > i \text{ or } k > j \}$ CFL
    - $\max(L2) = \{ \ \}$ (empty) Regular
    - $\min(L2) = \{ a^i b^j c^k \mid k = \min(i, j)+1 \}$ Non-CFL
- max(L1) shows CFL not closed under max
- min(L2) shows CFL not closed under min

Complement of ww

- Let L = { ww | w ∈ $\{a,b\}^+$ }. L is not a CFL
- Consider L's complement, it must be of form xayx'by' or xbyx'ay', where |x|=|x'| and |y|=|y'|
- The above reflects that this language has one "transcription error"
- This seems really hard to write a CFG but it's all a matter of how you view it
- We don't care about what precedes or follows the errors so long as the lengths are right
- Thus, we can view above as xax'yby' or xbx'y'ay', where |x|=|x'| and |y|=|y'|
- The grammar for this has rules
  S → AB | BA ; A → XAX | a ; B → XBX | b
  X → a | b

Solvable CFL Problems

- Let L be an arbitrary CFL generated by CFG G with start symbol S then the following are all decidable
  - Is w in L? — Run CKY; If S in final cell then w∈L
  - Is L empty (non-empty)? — Reduce G; If no rules left then empty
  - Is L finite (infinite)? — Reduce G; Run DFS(S); If no loops then finite

– Is w in L?	Run CKY If S in final cell then w∈L
– Is L empty (non-empty)?	Reduce G If no rules left then empty
– Is L finite (infinite)?	Reduce G Run DFS(S) If no loops then finite

Formalization of PDA

- $A = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$
- Q is finite set of states
- Σ is finite input alphabet
- Γ is finite set of stack symbols
- $\delta : Q \times \Sigma_e \times \Gamma_e \rightarrow 2^{Q \times \Gamma^*}$ is transition function
  - Note: Can limit stack push to Γ_e but it's equivalent!!
- $Z_0 \in \Gamma$ is an optional initial symbol on stack
- $F \subseteq Q$ is final set of states and can be omitted for some notions of a PDA

Notion of ID for PDA

- An instantaneous description for a PDA is [q, w, γ] where
  - q is current state
  - w is remaining input
  - γ is contents of stack (leftmost symbol is top)
- Single step derivation is defined by
  - $[q,ax,Z\alpha] \vdash [p,x,\beta\alpha]$ if $\delta(q,a,Z)$ contains (p,β)
- Multistep derivation ($\vdash^*$) is reflexive transitive closure of single step.

Language Recognized by PDA

- Given $A = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$
  there are three senses of recognition
- By final state
  $L(A) = \{w \mid [q_0,w,Z_0] \vdash^* [f,\lambda,\beta]\}$, where $f \in F$
- By empty stack
  $N(A) = \{w \mid [q_0,w,Z_0] \vdash^* [q,\lambda,\lambda]\}$
- By empty stack and final state
  $E(A) = \{w \mid [q_0,w,Z_0] \vdash^* [f,\lambda,\lambda]\}$, where $f \in F$

Top Down Parsing by PDA

- Given $G = (V, \Sigma, R, S)$, define
  $A = (\{q\}, \Sigma, \Sigma \cup V, \delta, q, S, \phi)$
- $\delta(q,a,a) = \{(q,\lambda)\}$ for all $a \in \Sigma$
- $\delta(q,\lambda,A) = \{(q,\alpha) \mid A \to \alpha \in R \text{ (guess)} \}$
- $N(A) = \mathscr{L}(G)$

- Give just one state, this is essentially stateless, except for stack

Top Down Parsing by PDA

$E \rightarrow E + T \mid T$

$T \rightarrow T * F \mid F$

$F \rightarrow (E) \mid Int$

- $\delta(q,+,+) = \{(q,\lambda)\}$, $\delta(q,*,*) = \{(q,\lambda)\}$,
- $\delta(q,Int,Int) = \{(q,\lambda)\}$,
- $\delta(q,(,() = \{(q,\lambda)\}$, $\delta(q,),)) = \{(q,\lambda)\}$
- $\delta(q,\lambda,E) = \{(q,E+T), (q,T)\}$
- $\delta(q,\lambda,T) = \{(q,T*F), (q,F)\}$
- $\delta(q,\lambda,F) = \{(q,(E)), (q,Int)\}$

Bottom Up Parsing by PDA

- Given $G = (V, \Sigma, R, S)$, define
  $A = (\{q,f\}, \Sigma, \Sigma \cup V \cup \{\$\}, \delta, q, \$, \{f\})$
- $\delta(q,a,\lambda) = \{(q,a)\}$ for all $a \in \Sigma$, SHIFT
- $\delta(q,\lambda,\alpha^R) \supseteq \{(q,A)\}$ if $A \rightarrow \alpha \in R$, REDUCE
  Cheat: looking at more than top of stack
- $\delta(q,\lambda,S) \supseteq \{(f,\lambda)\}$
- $\delta(f,\lambda,\$) = \{(f,\lambda)\}$, ACCEPT
- $E(A) = \mathscr{L}(G)$
- Could also do $\delta(q,\lambda,S\$) \supseteq \{(q,\lambda)\}$, $N(A) = \mathscr{L}(G)$

Bottom Up Parsing by PDA

$E \rightarrow E + T \mid T$

$T \rightarrow T * F \mid F$

$F \rightarrow (E) \mid Int$

- $\delta(q,+,\lambda)=\{(q,+)\}$, $\delta(q,*,\lambda)=\{(q,*)\}$, $\delta(q,Int,\lambda)=\{(q,Int)\}$, $\delta(q,(,\lambda)=\{(q,()\}$, $\delta(q,),\lambda)=\{(q,))\}$
- $\delta(q,\lambda,T+E) = \{(q,E)\}$, $\delta(q,\lambda,T) \supseteq \{(q,E)\}$
- $\delta(q,\lambda,F*T) \supseteq \{(q,T)\}$, $\delta(q,\lambda,F) \supseteq \{(q,T)\}$
- $\delta(q,\lambda,)E() \supseteq \{(q,F)\}$, $\delta(q,\lambda,Int) \supseteq \{(q,F)\}$
- $\delta(q,\lambda,E) \supseteq \{(f,\lambda)\}$
- $\delta(f,\lambda,\$) = \{(f,\lambda)\}$
- $E(A) = \mathscr{L}(G)$

Converting a PDA to CFG

- Book has one approach; here is another
- Let A = (Q, Σ, Γ, δ, q_0, Z, F) accept L by empty stack and final state
- Define A' = (Q$\cup${q_0',f}, Σ, $\Gamma\cup${\$}, δ', q_0', \$, {f}) where
  - δ'(q_0', λ, \$) = {(q_0, PUSH(Z)) or in normal notation {(q_0, Z\$)}
  - δ' does what δ does but only uses PUSH and POP instructions, always reading top of stack
    Note1: we need to consider using the \$ for cases of the original machine looking at empty stack, when using λ for stack check. This guarantees we have top of stack until very end.
    Note2: If original adds stuff to stack, we do pop, followed by a bunch of pushes.
  - We add (f, λ) = (f, POP) to δ'(q_f, λ, \$) whenever q_f is in F, so we jump to a fixed final state.
- Now, wlog, we can assume our PDA uses only POP and PUSH, has just one final state and accepts by empty stack and final state. We will assume the original machine is of this form and that its bottom of stack is \$.
- Define G = (V, Σ, R, S) where
  - V = {S} $\cup$ { <q, X, p> | q,p $\in$ Q, X $\in$ Γ }
  - R on next page

Rules for PDA to CFG

- R contains rules as follows:
  $S \rightarrow <q_0,\$,f>$ where $F = \{f\}$
  meaning: want to generate w whenever
  $[q_0,w,\$] \vdash^* [f,\lambda,\lambda]$
- Remaining rules are:
  $<q,X,p> \rightarrow a<s,Y,t><t,X,p>$
  whenever $\delta(q,a,X) \supseteq \{(s,PUSH(Y))\}$
  $<q,X,p> \rightarrow a$
  whenever $\delta(q,a,X) \supseteq \{(p,POP)\}$
- Want $<q,X,p> \Rightarrow^* w$ when $[q,w,X] \vdash^* [p,\lambda,\lambda]$

Greibach Normal Form

- Each rule of a GNF is constrained to be of form:
  $A \to a\alpha$, $A \in V$, $a \in \Sigma$, $\alpha \in V^*$
- If the language contains λ then we allow
  $S \to \lambda$
  and constrain S to not be on right hand side of any rule
- The beauty of this form is that, in a bottom up parse, every step consumes an input character and so parse is linear (if we guess right)
- We will not show details of conversion but it is dependent on starting in CNF and then removing left recursion, both of which we have already shown

Closure Properties

Context Free Languages

Intersection with Regular

- CFLs are closed under intersection with Regular sets
  - To show this we use the equivalence of CFGs generative power with the recognition power of PDAs.
  - Let $A_0 = (Q_0, \Sigma, \Gamma, \delta_0, q_0, \$, F_0)$ be an arbitrary PDA
  - Let $A_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ be an arbitrary DFA
  - Define $A_2 = (Q_0 \times Q_1, \Sigma, \Gamma, \delta_2, <q_0,q_1> \$, F_0 \times F_1)$ where
    - $\delta_2(<q,s>, a, X) \supseteq \{(<q',s'>, \alpha)\}$, $a \in \Sigma \cup \{\lambda\}$, $X \in \Gamma$ iff
      $\delta_0(q, a, X) \supseteq \{(q', \alpha)\}$ and
      $\delta_1(s,a) = s'$ (if $a=\lambda$ then $s' = s$).
  - Using the definition of derivations we see that
    $[<q_0,q_1>, w, \$] \vdash^* [<t,s>, \lambda, \beta]$ in A_2 iff
    $[q_0, w, \$] \vdash^* [t, \lambda, \beta]$ in A_0 and
    $[q_1, w] \vdash^* [s, \lambda]$ in A_1
    But then $w \in \mathcal{F}(A_2)$ iff $t \in F_0$ and $s \in F_1$ iff $w \in \mathcal{F}(A_0)$ and $w \in \mathcal{F}(A_1)$

Substitution

- CFLs are closed under CFL substitution
  - Let $G=(V,\Sigma,R,S)$ be a CFG.
  - Let f be a substitution over Σ such that
    - $f(a) = L_a$ for $a \in \Sigma$
    - $G_a = (V_a,\Sigma_a,R_a,S_a)$ is a CFG that produces L_a.
    - No symbol appears in more than one of V or any V_a
  - Define $G_f = (V \cup_{a\in\Sigma}V_a, \cup_{a\in\Sigma}\Sigma_a, R' \cup_{a\in\Sigma}R_a, S)$
    - $R' = \{ A \rightarrow g(\alpha)$ where $A \rightarrow \alpha$ is in $R \}$
    - $g: (V\cup\Sigma)^* \rightarrow (V \cup_{a\in\Sigma}S_a)^*$
    - $g(\lambda) = \lambda$; $g(B) = B$, $B \in V$; $g(a) = S_a$, $a \in \Sigma$
    - $g(\alpha X) = g(\alpha)\, g(X)$, $|\alpha| > 0$, $X \in V\cup\Sigma$
  - Claim, $f(\mathscr{L}(G)) = \mathscr{L}(G_f)$, and so CFLs closed under substitution and homomorphism.

More on Substitution

- Consider G'_f. If we limit derivations to the rules $R' = \{ A \to g(\alpha)$ where $A \to \alpha$ is in $R \}$ and consider only sentential forms over the $\cup_{a \in \Sigma} S_a$, then $S \Rightarrow^* S_{a1}\ S_{a2} \ldots S_{an}$ in G' iff $S \Rightarrow^*$ a1 a2 … an iff a1 a2 … an $\in \mathcal{L}(G)$. But, then $w \in \mathcal{L}(G)$ iff $f(w) \in \mathcal{L}(G_f)$ and, thus, $f(\mathcal{L}(G)) = \mathcal{L}(G_f)$.

- Given that CFLs are closed under intersection, substitution, homomorphism and intersection with regular sets, we can recast previous proofs to show that CFLs are closed under
  - Prefix, Suffix, Substring, Quotient with Regular Sets

- Later we will show that CFLs are <u>not</u> closed under Quotient with CFLs.

Context Sensitive

Context Sensitive Grammar

G = (V, Σ, R, S) is a PSG where

Each member of R is a rule whose right side is no shorter than its left side.

The essential idea is that rules are length preserving, although we do allow $S \rightarrow \lambda$ so long as S never appears on the right hand side of any rule.

A context sensitive grammar is denoted as a CSG and the language generated is a Context Sensitive Language (CSL).

The recognizer for a CSL is a Linear Bounded Automaton (LBA), a form of Turing Machine (soon to be discussed), but with the constraint that it is limited to moving along a tape that contains just the input surrounded by a start and end symbol.

Phrase Structured Grammar

We previously defined PSGs. The language generated by a PSG is a Phrase Structured Language (PSL) but is more commonly called a recursively enumerable (re) language. The reason for this will become evident a bit later in the course.

The recognizer for a PSL (re language) is a Turing Machine, a model of computation we will soon discuss.

CSG Example#1

$L = \{ a^n b^n c^n \mid n>0 \}$

$G = (\{A,B,C\}, \{a,b,c\}, R, A)$ where R is

$A \rightarrow aBbc \mid abc$

$B \rightarrow aBbC \mid abC$

Note: $A \Rightarrow aBbc \Rightarrow n\ a^{n+1}(bC)^n\ bc$ // $n>0$

$Cb \rightarrow bC$ // Shuttle C over to a c

$Cc \rightarrow cc$ // Change C to a c

Note: $a^{n+1}(bC)^n\ bc \Rightarrow^* a^{n+1}b^{n+1}c^{n+1}$

Thus, $A \Rightarrow^* a^n b^n c^n$, $n>0$

CSG Example#2

L = { ww | w ∈{0,1}$^+$ }

G = ({S,A,X,Z,<0>,<1>}, {0,1}, R, S) where R is

S → 00 | 11 | 0A<0> | 1A<1>

A → 0AZ | 1AX | 0Z | 1X

Z0 → 0Z	Z1 → 1Z	// Shuttle Z (for owe zero)
X0 → 0X	X1 → 1X	// Shuttle X (for owe one)
Z<0> → 0<0>	Z<1> → 1<0>	// New 0 must be on rhs of old 0/1’s
X<0> → 0<1>	X<1> → 1<1>	// New 1 must be on rhs of old 0/1’s
<0> → 0		// Guess we are done
<1> → 1		// Guess we are done