**COT 6410 Spring2019 Sample Final Exam**

1. Let set **A** be **infinite recursive**, **B** be **re non-recursive** and **C** be **non-re**. Using the terminology **(REC) recursive**, **(RE) re, non-recursive**, **(NR) non-re (possibly co-re)**, categorize each set by dealing with the cases I present, saying whether or not the set can be of the given category and briefly, but convincingly, justifying each answer (BE COMPLETE). You may assume sets like $\aleph$ are infinite REC; **K** and $\mathbf{K_0}$ are RE; and **TOTAL** is non-re. You may also assume, for any set **S**, the existence of comparably hard sets
$S_E = \{2x \mid x \in S\}$ and $S_D = \{2x+1 \mid x \in S\}$.

   **a.)** $A + B = \{ x \mid x = y + z, \text{ for some } y \in A \text{ and some } z \in B \}$

   **REC:** $A = \aleph$, $B = K_E$, $A+B = \{ x \mid x \geq \min y \in K_E \}$.

   **This is the complement of a finite set and is hence decidable as the finite set is.**

   **b.)** $A \cap C = \{ x \mid x \in A \text{ and } x \in C \text{ and } x \notin A \}$

   **RE:** $A = E = \{2x \mid x \in \aleph\}$, $C = TOTAL_D \cup K_E$.

   $A \cap C = K_E$ **which is RE.**

2. Choosing from among **(REC) recursive**, **(RE) re non-recursive, (coRE) co-re non-recursive**, **(NRNC) non-re/non-co-re**, categorize each of the sets in a) through d). Justify your answer by showing some minimal quantification of some known recursive predicate.

   **a)** $A = \{ \langle f,g \rangle \mid \exists x\ \varphi_f(x)\downarrow \text{ and } \varphi_g(x) = \varphi_f(x) \}$.

   *$\exists\langle x,t\rangle$ [STP(f,x,t) & STP(g,x,t) & Value(f,x,t) = Value(g,x,t)]* ______ *RE*

   **b.)** $B = \{ f \mid \text{range}(\varphi_f) \text{ is empty} \}$

   *$\forall\langle x,t\rangle$ [~STP(f,x,t)]* ______ *co-RE*

   **c.)** $C = \{ \langle f,x \rangle \mid \varphi_f(x)\downarrow \text{ but takes at least 10 steps to do so} \}$

   *$\exists t$ [STP(f,x,t) & ~STP(f,x,9)]* ______ *RE*

   **d.)** $D = \{ f \mid \varphi_f \text{ diverges for some value of } x \}$

   *$\exists x\ \forall t$ [~STP(f,x,t) ]* ______ *NRNC*

3. Looking back at Question 1, which of these are candidates for using Rice's Theorem to show their unsolvability? Check all for which Rice Theorem might apply.

   **a)** ____√ **b)** ____√ **c)** ____ **d)** ____√

4. Let **S** be an arbitrary semi-decidable set. By definition, **S** is the domain of some partial recursive function $\mathbf{g_S}$. Using $\mathbf{g_S}$, constructively show that **S** is the range of some partial recursive function, $\mathbf{f_S}$. No proof is required; just the construction is needed here.

   $f_S(x) = x * \exists t\ [\ STP(x, g_s, t)\ ]$ **or**

   $f_S(x) = x * (g_s(x) - g_s(x) + 1)$

**5.** Using the definition that **S** is recursively enumerable iff **S** is the range of some effective procedure $\mathbf{f_S}$ (partial recursive function), prove that if both **S** and its complement **~S** are recursively enumerable (using semi-decision effective procedures $\mathbf{f_S}$ and $\mathbf{f_{\sim S}}$) then **S** is decidable. To get full credit, you must show the characteristic function for S, $\chi_S$, in all cases. Also, be sure to discuss why your $\chi_S$ works.

***Define** $\chi_S(x) = f_S(\mu\langle y, t\rangle[(STP(f_S,y,t)\ \&\ (VALUE(f_S,y,t)= x))$ or*

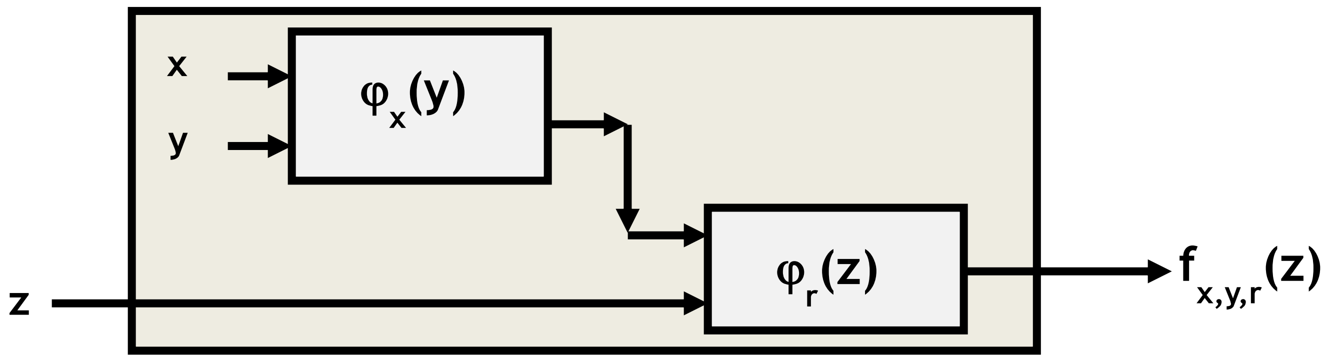
*$(STP(f_{\sim S},y,t)\ \&\ (VALUE(f_{\sim S},y,t)= x))\ ])$*

***If** $x \in S$ **then** $\exists y\ f_S(y) = x$ **and so** $\chi_S(x) = 1$ **(true)***

***If** $x \notin S$ **then** $\exists y\ f_{\sim S}(y) = x$ **and so** $\chi_S(x) = 0$ **(false)***

***Thus,** $\chi_S(x)$ **meets our requirements.***

**6**. Rice's Theorem deals with attributes of certain types of problems **P** about partial recursive functions and their corresponding sets of indices $\mathbf{S_P}$. The following image describing a function $\mathbf{f_{x,y,r}}$ is central to understanding Rice's Theorem.




Explain the meaning of this by indicating:

**a.)** What assumption do we make about what kind of functions are not in P?
**We assume no function with empty domain has property P.**

**b.)** What is **r**, how is it chosen and how can we guarantee its existence?
**r is the index of some function with property P. One must exist since P is non-trivial.**

**c.)** Using recursive function notations, write down precisely what $\mathbf{f_{x,y,r}}$ computes for the Strong Form of Rice's Theorem.
$f_{x,y,r}(z) = \varphi_x(y) - \varphi_x(y) + \varphi_r(z)$

How does this function $\mathbf{f_{x,y,r}}$ behave with respect to **x**,**y** and **r**, and how does that relate to the original problem, **P**, and set, $\mathbf{S_P}$?

**If** $\varphi_x(y)\downarrow$ **then** $f_{x,y,r}(z) = \varphi_r(z)\ \forall z$ **and** $f_{x,y,r} \in S_P$.

**If** $\varphi_x(y)\uparrow$ **then** $f_{x,y,r}(z)\uparrow\ \forall z$ **and** $f_{x,y,r} \notin S_P$.
**Thus, we could decide the halting problem if we could decide membership in** $S_P$**, so P is an undecidable problem.**

**7.** Define **NAT = { f | range(f) = ℵ }**. That is, **f ∈NAT** iff **f**'s range includes every natural number.

**a.)** Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of **NAT**.

*∀k ∃<x, t> [STP(f,x,t) && (Value(f,x,t) == k)]*

**b.)** Use Rice's Theorem to prove that **NAT** is undecidable.

***First, NAT is non-trivial as the identity, I(x)=x, is in NAT and the Constant Zero, Z(x)=0, is not.***

***Second, let f and g be arbitrary indices of arbitrary effective procedures, such that range($\varphi_f$) = range($\varphi_g$).***

***f is in NAT iff range($\varphi_f$) = ℵ iff range($\varphi_f$) = ℵ.***

***This means NAT satisfies both properties of the weak form of Rice's Theorem associated with ranges and is therefore undecidable.***

**c.)** Show that **TOT $\leq_m$ NAT**, where **TOT = { f | ∀x $\varphi_f$(x)↓ }**.

***Let f be arbitrary. Define an algorithmic mapping $G_f$ from indices to indices as $G_f$ (x) = f(x)-f(x)+x.***

***Now, $G_f$ (x) = I(x) (the Identity function) iff f ∈ TOTAL and***
***∃x x∉range($G_f$) iff f ∉ TOTAL This will be any x where $\varphi_f$(x)↑.***
***Thus, f is in TOT iff $G_f$ is in NAT. Thus, TOTAL $\leq_m$ NAT.***

**8.** Why does Rice's Theorem have nothing to say about the following? Explain by showing some condition of Rice's Theorem that is not met by the stated property.
**AT-LEAST-LINEAR = { f | ∀y $\varphi_f$(y) converges in no fewer than y steps }**.
**We can deny the 2nd condition of Rice's Theorem since**
**Z, where Z(x) = 0, implemented by the TM R converges in one step no matter what x is and hence is not in AT-LEAST-LINEAR**

**Z', defined by the TM $\mathcal{L}\ \mathcal{R}$ R, is in AT-LEAST-LINEAR**

**However, ∀x [ Z(x) = Z'(x) ], so they have the same I/O behavior and yet one is in and the other is out of AT-LEAST-LINEAR, denying the 2nd condition of Rice's Theorem**

**9.** Consider the following set of independent tasks with associated task times:
**(T1,4), (T2,5), (T3,2), (T4,7), (T5,1), (T6,4), (T7,8)**
Fill in the schedules for these tasks under the associated strategies below.

Greedy using the list order above:

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| T1 | T1 | T1 | T1 | T3 | T3 | T5 | T6 | T6 | T6 | T6 | T7 | T7 | T7 | T7 | T7 | T7 | T7 | T7 | | | |
| T2 | T2 | T2 | T2 | T2 | T4 | T4 | T4 | T4 | T4 | T4 | T4 | | | | | | | | | | |

Greedy using a reordering of the list so that longest running tasks appear earliest in the list:

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| T7 | T7 | T7 | T7 | T7 | T7 | T7 | T7 | T1 | T1 | T1 | T1 | T6 | T6 | T6 | T6 | | | | | | |
| T4 | T4 | T4 | T4 | T4 | T4 | T4 | T2 | T2 | T2 | T2 | T2 | T3 | T3 | T5 | | | | | | | |

**10.** We described the proof that **3SAT** is polynomial reducible to **Subset-Sum**. You must repeat that.

**a.)** Assuming a **3SAT** expression **(a + a + ~b) (~a + b + c)**, fill in all omitted values (zeroes elements can be left as omitted) of the reduction from **3SAT** to **Subset-Sum**.

| | **a** | **b** | **c** | **a + a + ~b** | **~a + b + c** |
|---|---|---|---|---|---|
| **a** | 1 | | | 1 or 2 | |
| **~a** | 1 | | | | 1 |
| **b** | | 1 | | | 1 |
| **~b** | | 1 | | 1 | |
| **c** | | | 1 | | 1 |
| **~c** | | | 1 | | |
| **C1** | | | | 1 | |
| **C1'** | | | | 1 | |
| **C2** | | | | | 1 |
| **C2'** | | | | | 1 |
| | **1** | **1** | **1** | **3** | **3** |

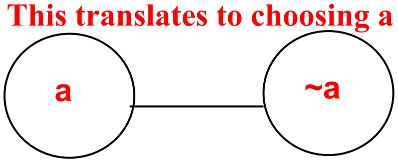
**b.)** List some subset of the numbers above (each associated with a row) that sums to **1 1 1 3 3**. Indicate what the related truth values are for **a**, **b** and **c**.

**a = T ; b = T ; c = T**

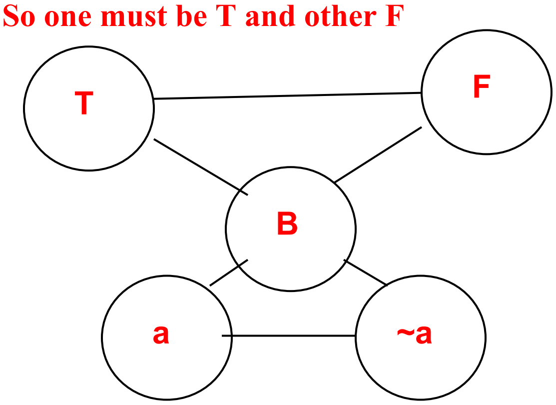
| | | |
|---|---|---|
| **1 0 0 1 0** | **or** | **1 0 0 2 0** |
| **0 1 0 0 1** | | **0 1 0 0 1** |
| **0 0 1 0 1** | | **0 0 1 0 1** |
| **0 0 0 1 0** | | **0 0 0 1 0** |
| **0 0 0 1 0** | | **0 0 0 0 1** |
| **0 0 0 0 1** | | |

**11.** Present a gadget used in the reduction of **3-SAT** to some graph theoretic problem where the gadget guarantees that each variable is assigned either **True** or **False**, but not both. Of course, you must tell me what graph theoretic problem is being shown **NP-Complete** and you must explain why the gadget works.

**Vertex Cover**
**Must Cover each Edge**
**Set goal to min vertices**
**Must choose one but not both are needed**
**This translates to choosing a or ~a**




**3-Color**
**Cannot choose B for either a or ~a**
**So one must be T and other F**



**12.** Let **Q** be some problem (an optimization or decision problem). Assuming **≤ p** means many-one reducible in polynomial time and **≤ tp** means Turing-reducible in polynomial time, categorize **Q** as being in one of **P**, **NP**, **co-NP**, **NP-Complete**, **NP-Easy**, **NP-Hard**, or **NP-Equivalent** (see first two pages for definitions of each of these concepts). For each case, choose the most precise category. I filled in one answer already.

| **Description of Q** | **Category** |
|---|---|
| **Q** is decidable in deterministic polynomial time | **P** |
| For some **R** in **NP**, **Q ≤ tp R** | **NP-Easy** |
| **Q** is both **NP-Easy** and **NP-Hard** | **NP-Equivalent** |
| **Q** is in **NP** and if **R** is in **NP** then **R ≤ p Q** | **NP-Complete** |
| A solution to **Q** is verifiable in deterministic polynomial time | **NP** |
| **Q**'s complement is in **NP** | **Co-NP** |

**13.** A graph **G** is **k-Colorable** if its vertices can be colored using just **k** (or fewer colors) such that adjacent vertices have different colors. The **Chromatic Number** of a graph **G** is the smallest number **k** for which **G** is **k-Colorable**. **k-Colorable** is a decision problem that has parameters (**G**, **k**), whereas the **Chromatic Number** problem is a function with a single parameter **G**. In all cases, assume **G** has **n** vertices.

**a.)** Show that **k-Colorable ≤ tp Chromatic Number** (**≤ tp** means Turing reducible in polynomial time).

**G is k-Colorable iff its Chromatic Number is some j ≤ k**

**This can be checked by just one invocation of the Oracle for Chromatic Number**

**b.)** Show that **Chromatic Number ≤ tp k-Colorable** (**≤ tp** means Turing reducible in polynomial time).

**G's Chromatic Number is no worse than n, the number of vertices. Doing a binary search, we can make at most $\log_2 n$ calls to the oracle for k-Colorable to determine the smallest number for which G is k-colorable**

**14. Partition** refers to the decision problem as to whether some set of positive integers **S** can be partitioned into two disjoint subsets whose elements have equal sums. **Subset-Sum** refers to the decision problem as to whether there is a subset of some set of positive integers **S** that precisely sums to some goal number **G**.

**a.)** Show that **Partition ≤p Subset-Sum**.

***Look at notes***

**b.)** Show that **Subset-Sum ≤p Partition**.

***Look at notes***

**15.** **QSAT** is the decision problem to determine if an arbitrary fully quantified Boolean expression is true. Note: **SAT** only uses existential, whereas **QSAT** can have universal qualifiers as well so it includes checking for Tautologies as well as testing Satisfiability. What can you say about the complexity of **QSAT** (is it in **P**, **NP**, **NP-Complete**, **NP-Hard**)? Justify your conclusion.

**QSAT is NP-Hard. This is so since SAT trivially reduces to QSAT (it is a subproblem of QSAT). Since SAT is known to be NP-Complete then some NP-Complete problem polynomially reduces to QSAT. This makes QSAT NP-Hard. As we cannot (at least not yet) show QSAT is in NP, then NP-Hard is the best we can do.**

**16.** Specify True (**T**) or False (**F**) for each statement.

| **Statement** | **T or F** |
|---|---|
| Every Regular Language is also a Context Free Language | ***T*** |
| Phrase Structured Languages are the same as RE Languages | ***T*** |
| The Context Free Languages are closed under Complement | ***F*** |
| A language is recursive iff it and its complement are re | ***T*** |
| PCP is undecidable even for one letter systems | ***F*** |
| Membership in Context Sensitive Languages is undecidable | ***F*** |
| Every RE language is Turing reducible to its complement | ***T*** |
| Emptiness is undecidable for Context Sensitive Languages | ***T*** |
| The complement of a trace language is Context Free | ***T*** |
| The word problem for two-letter Semi-Thue Systems is decidable | ***F*** |