COT6410 Spring 2019 Assignment #4 Sample

Define **NTR = NON_TRIVIAL_RANGE = (f | |range(f)| > 1 }**.

a.) Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of this set. (Hint: Look at **c.)** and **d.)** to get a clue as to what this must be.)

b.) Use Rice's Theorem to prove that **NTR** is undecidable.

c.) Show that $\mathbf{K_0 \leq_m}$ **NTR**, where $\mathbf{K_0 = \{ \langle x,y \rangle \mid \varphi_x(y)\downarrow \}}$.

d.) Show that **NTR** $\mathbf{\leq_m K_0}$.

e.) From **a.)** through **d.)** what can you conclude about the complexity of **NTR**
(Recursive, RE, RE-COMPLETE, CO-RE, CO-RE-COMPLETE, NON-RE/NON-CO-RE)?

COT6410 Spring 2019 Assignment #4 Sample with Solutions

Define **NTR = NON_TRIVIAL_RANGE = (f | |range(f)| > 1 }**.

a.) Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of this set. (Hint: Look at **c.)** and **d.)** to get a clue as to what this must be.)

$\exists \langle x,y,t \rangle\ [STP(f,x,t)\ \&\&\ STP(f,y,t)\ \&\&\ (VALUE(f,x,t) \neq VALUE(f,y,t))]$

b.) Use Rice's Theorem to prove that **NTR** is undecidable.

First show NTR is non-trivial:

$I(x) = x \in NTR \qquad C_0(x) = 0 \notin NTR$
and thus, the set and its complement are non-empty as required.

Let f, g be two arbitrary indices (functions) such that range (φ_f ***) = range (*** φ_g ***).***

$f \in NTR$ iff $\lvert range(\varphi_f)\rvert > 1$	***Definition of NTR***
iff $\lvert range(\varphi_g)\rvert > 1$	***Since we assumed range (*** φ_f ***) = range (*** φ_g ***)***
iff $g \in NTR$	***Definition of NTR***

This weak form of Rice's Theorem shows NTR to be undecidable.

c.) Show that $\mathbf{K_0 \le_m NTR}$, where $\mathbf{K_0} = \{ \langle x,y \rangle \mid \varphi_x(y)\downarrow \}$.

Let $\langle x,y \rangle$ ***be an arbitrary pair of natural numbers.***
Define $f_{x,y}(z) = \varphi_x(y) - \varphi_x(y) + z$

$\langle x,y \rangle \in K_0$ ***implies*** $\forall z\ f_{x,y}(z) = z$ ***implies Range(*** $f_{x,y}$ ***) is infinite implies*** $f_{x,y} \in NTR$

$\langle x,y \rangle \notin K_0$ ***implies*** $\forall z\ f_{x,y}(z)\uparrow$ ***implies Range(*** $f_{x,y}$ ***) is empty implies*** $f_{x,y} \notin NTR$

Thus, $\langle x,y \rangle \in K_0 \Leftrightarrow f_{x,y} \in NTR$

And so, $K_0 \le_m NTR$

d.) Show that $\mathbf{NTR \le_m K_0}$.

Let f be an arbitrary index (function)
Define $g_f(z) = \exists \langle x,y,t \rangle\ [STP(f,x,t)\ \&\&\ STP(f,y,t)\ \&\&\ (VALUE(f,x,t) \neq VALUE(f,y,t))]$

$f \in NTR$ ***implies*** $\forall z\ g_f(z) = 1$ ***implies*** $\langle g_f, 0 \rangle \in K_0$

$f \notin NTR$ ***implies*** $\forall z\ g_f(z)\uparrow$ ***implies*** $\langle g_f, 0 \rangle \notin K_0$

Thus, $f \in NTR \Leftrightarrow \langle g_f, 0 \rangle \in K_0$

And so, $NTR \le_m K_0$

e.) From **a.)** through **d.)** what can you conclude about the complexity of **NTR**
(Recursive, RE, RE-COMPLETE, CO-RE, CO-RE-COMPLETE, NON-RE/NON-CO-RE)?

RE-COMPLETE