Define $NTR = \text{NON\_TRIVIAL\_RANGE} = \{ f \mid |\text{range}(f)| > 1 \}$.

a.) Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of this set. (Hint: Look at c.) and d.) to get a clue as to what this must be.)

b.) Use Rice’s Theorem to prove that $NTR$ is undecidable.

c.) Show that $K_0 \leq_m NTR$, where $K_0 = \{ <x,y> \mid \phi_x(y) \downarrow \}$.

d.) Show that $NTR \leq_m K_0$.

e.) From a.) through d.) what can you conclude about the complexity of $NTR$ (Recursive, RE, RE-COMPLETE, CO-RE, CO-RE-COMPLETE, NON-RE/NON-CO-RE)?
Define \( NTR = \text{NON_TRIVIAL}_\text{RANGE} = \{ f | |\text{range}(f)| > 1 \}. \)

a.) Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of this set. (Hint: Look at c.) and d.) to get a clue as to what this must be.)

\[ \exists <x,y,t> [\text{STP}(f,x,t) \& \& \text{STP}(f,y,t) \& \& (\text{VALUE}(f,x,t) \neq \text{VALUE}(f,y,t))] \]

b.) Use Rice’s Theorem to prove that \( NTR \) is undecidable.

First show \( NTR \) is non-trivial:

\[
I(x) = x \in NTR \quad C_0(x) = 0 \notin NTR \quad \text{and thus, the set and its complement are non-empty as required.}
\]

Let \( f, g \) be two arbitrary indices (functions) such that range (\( \varphi_f \)) = range (\( \varphi_g \)).

\[
f \in NTR \iff |\text{range (} \varphi_f \text{)}| > 1 \text{ \quad Definition of } NTR
\]

\[\begin{align*}
&\iff |\text{range (} \varphi_g \text{)}| > 1 \quad \text{Since we assumed range (} \varphi_f \text{) = range (} \varphi_g \text{)} \\
&\iff g \in NTR \quad \text{Definition of } NTR
\end{align*}\]

This weak form of Rice’s Theorem shows \( NTR \) to be undecidable.

c.) Show that \( K_0 \leq_m NTR \), where \( K_0 = \{ <x,y> | \varphi_x(y) \downarrow \} \).

Let \( <x,y> \) be an arbitrary pair of natural numbers.

Define \( f_{x,y}(z) = \varphi_x(y) - \varphi_x(y) + z \)

\[
<x,y> \in K_0 \implies \forall z \ f_{x,y}(z) = z \implies \text{Range}(f_{x,y}) \text{ is infinite implies } f_{x,y} \in NTR
\]

\[
<x,y> \notin K_0 \implies \forall z \ f_{x,y}(z) \uparrow \implies \text{Range}(f_{x,y}) \text{ is empty implies } f_{x,y} \notin NTR
\]

Thus, \( <x,y> \in K_0 \iff f_{x,y} \in NTR \)

And so, \( K_0 \leq_m NTR \)

d.) Show that \( NTR \leq_m K_0 \).

Let \( f \) be an arbitrary index (function)

Define \( g_f(z) = \exists <x,y,t> [\text{STP}(f,x,t) \& \& \text{STP}(f,y,t) \& \& (\text{VALUE}(f,x,t) \neq \text{VALUE}(f,y,t))] \)

\[
f \in NTR \implies \forall z \ g_f(z) = 1 \implies <g_f,0> \in K_0
\]

\[
f \notin NTR \implies \forall z \ g_f(z) \uparrow \implies <g_f,0> \notin K_0
\]

Thus, \( f \in NTR \iff <g_f,0> \in K_0 \)

And so, \( NTR \leq_m K_0 \)

e.) From a.) through d.) what can you conclude about the complexity of \( NTR \) (Recursive, \( \text{RE} \), \( \text{RE-COMPLETE} \), \( \text{CO-RE} \), \( \text{CO-RE-COMPLETE} \), \( \text{NON-RE/NON-CO-RE} \))?

\( \text{RE-COMPLETE} \)