Assignment#4 Key
1. NotDominating(ND) = \{ f \mid \text{for some } x, f(x) < x \}.

a.) Show some minimal quantification of some known primitive recursive predicate that provides an upper bound for the complexity of ND.

\[ \exists \langle x, t \rangle \ [\text{STP}(f, x, t) \land (\text{VALUE}(f, x, t) < x)] \]

b.) Use Rice’s Theorem to prove that ND is undecidable. Be Complete.

ND is non-trivial as \( C_0(x) = 0 \in \text{ND} \) and \( S(x) = x+1 \notin \text{ND} \)

Let \( f, g \) be two arbitrary indices of procedures such that \( \forall x \ f(x) = g(x) \)

\( f \in \text{ND} \) iff \( \exists x \ f(x) < x \) iff \( f(x_0) < x_0 \) for some \( x_0 \) iff \( g(x_0) < x_0 \) as \( \forall x \ f(x) = g(x) \) implies \( \exists x \ g(x) < x \) iff \( g \in \text{ND} \)

\( f \notin \text{ND} \) iff \( \forall x \ [f(x) \downarrow \implies f(x) > x] \) iff \( \forall x \ [g(x) \downarrow \implies g(x) > x] \) as \( \forall x \ f(x) = g(x) \) iff \( g \notin \text{ND} \)

c.) Show that \( K = \{ f \mid f(f) \text{ converges} \} \) is many-one reducible to ND.

Let \( f \) be an arbitrary index. From \( f \), define \( \forall x \ F_f(x) = f(f) - f(f) \).

\( f \in K \) implies \( \forall x \ F_f(x) = 0 \) implies \( F_f \in \text{ND} \).

\( f \notin K \) implies \( \forall x \ F_f(x) \text{ diverges} \) implies \( F_f \notin \text{ND} \).

Thus, \( K \leq_m \text{ND} \)

d.) Show that ND is many-one reducible to \( K = \{ f \mid f(f) \text{ converges} \} \)

Let \( f \) be an arbitrary index. From \( f \), define \( \forall y \ F_f(y) = \exists \langle x, t \rangle \ \text{STP}(f, x, t) \land (\text{VALUE}(f, x, t) < x) \)

\( f \in \text{ND} \) implies \( \forall y \ F_f(y) \text{ converges} \) implies \( F_f(F_f) \text{ converges} \) implies \( F_f \in K \)

\( f \notin \text{ND} \) implies \( \forall y \ F_f(y) \text{ diverges} \) implies \( F_f \notin K \).

Thus, \( \text{ND} \leq_m K \)
2. **AlwaysDominates**\((AD) = \{ f \mid \text{for all } x, f(x) > x \} \)

a.) Show some minimal quantification of some known primitive recursive predicate that provides an upper bound for the complexity of \(AD\).

\[ \forall x \exists t \left[ \text{STP}(f,x,t) \& (\text{VALUE}(f,x,t) > x) \right] \]

b.) Use Rice's Theorem to prove that \(AD\) is undecidable. Be Complete.

\(AD\) is non-trivial as \(S(x) = x+1 \in AD\) and \(C0(x) = 0 \notin AD\)

Let \(f,g\) be two arbitrary indices of procedures such that \(\forall x f(x) = g(x)\)

\(f \in AD \text{ iff } \forall x f(x) < x \text{ iff } \forall x g(x) < x \text{ iff } g \in ND\)

c.) Show that \(TOT = \{ f \mid \text{for all } x, f(x) \text{ converges } \}\) is many-one reducible to \(AD\).

Let \(f\) be an arbitrary index. From \(f\), define \(\forall x F_f(x) = f(x) - f(x) + x + 1\).

\(f \in TOT \text{ implies } \forall x F_f(x) = x+1 \text{ implies } F_f \in AD\).

\(f \notin TOT \text{ implies } \exists x F_f(x) \text{ diverges implies } F_f \notin AD\).

Thus, \(TOT \leq_m AD\)

d.) Show that \(AD\) is many-one reducible to \(TOT = \{ f \mid \text{for all } x, f(x) \text{ converges } \}\)

Let \(f\) be an arbitrary index. From \(f\), define \(\forall x F_f(x) = \mu y \left[ f(x) > x \right]\)

\(f \in AD \text{ implies } \forall x F_f(x) \text{ converges implies } F_f \in TOT\)

\(f \notin AD \text{ implies } \exists x F_f(x) \text{ diverges implies } F_f \notin TOT\).

Thus, \(AD \leq_m TOT\)