**Generally useful information.**

* The notation **z =** **<x,y>** denotes the pairing function with inverses **x =** **<z>1** and **y =** **<z>2**.
* The minimization notation **μ y [P(…,y)]** means the least **y** (starting at **0**) such that **P(…,y)** is true. The bounded minimization (acceptable in primitive recursive functions) notation   
  **μ y (u≤y≤v) [P(…,y)]** means the least **y** (starting at **u** and ending at **v**) such that **P(…,y)** is true. Unlike the text, I find it convenient to define **μ y (u≤y≤v) [P(…,y)]** to be **v+1**, when no **y** satisfies this bounded minimization.
* The tilde symbol, **~,** means the complement. Thus, set **~S** is the set complement of set **S**, and predicate **~P(x)** is the logical complement of predicate **P(x).**
* A function **P** is a predicate if it is a logical function that returns either **1** (**true**) or **0** (**false**). Thus, **P(x)** means **P** evaluates to **true** on **x**, but we can also take advantage of the fact that **true** is **1** and **false** is **0** in formulas like **y × P(x)**, which would evaluate to either **y** (if **P(x)**) or **0** (if **~P(x)**).
* A set **S** is recursive if **S** has a total recursive characteristic function **χS**, such that **x ∈ S ⇔ χS(x)**. Note **χS** is a predicate. Thus, it evaluates to **0** (**false**), if **x ∉ S**.
* When I say a set **S** is re, unless I explicitly say otherwise, you may assume any of the following equivalent characterizations:

1. **S** is either empty or the range of a total recursive function **fS**.
2. **S** is the domain of a partial recursive function **gS**.

* If I say a function **g** is partially computable, then there is an index **g** (I know that’s overloading, but that’s okay as long as we understand each other), such that **Φg(x) = Φ(x, g) = g(x)**. Here **Φ** is a universal partially recursive function.   
  Moreover, there is a primitive recursive function **STP**, such that   
  **STP(g, x, t)** is **1** (true), just in case **g**, started on **x**, halts in **t** or fewer steps.   
  **STP(g, x, t)** is **0** (false), otherwise.   
  Finally, there is another primitive recursive function **VALUE**, such that   
  **VALUE(g, x, t)** is **g(x)**, whenever **STP(g, x, t)**.   
  **VALUE(g, x, t)** is defined but meaningless if **~STP(g, x, t)**.
* The notation **f(x)↓** means that **f** converges when computing with input **x**, but we don’t care about the value produced. In effect, this just means that **x** is in the domain of **f**.
* The notation **f(x)↑** means **f** diverges when computing with input **x**. In effect, this just means that **x** is **not** in the domain of **f**.
* The **Halting Problem** for any effective computational system is the problem to determine of an arbitrary effective procedure **f** and input **x**, whether or not **f(x)↓**. The set of all such pairs, **K0**, is a classic re non-recursive one.
* The **Uniform Halting Problem** is the problem to determine of an arbitrary effective procedure **f**, whether or not **f** is an algorithm (halts on all input). The set of all such function indices is a classic non re one.
* **A ≤m B** (**A** many-one reduces to **B**) means that there exists a total recursive function **f** such that   
  **x ∈ A ⇔ f(x) ∈ B**. If **A ≤m B** and **B ≤m A** then we say that **A ≡m B** (**A** is many-one equivalent to **B)**. If the reducing function is 1-1, then we say **A ≤1 B** (**A** one-one reduces to **B**) and **A ≡1 B** (**A** is one-one equivalent to **B**).

**COT 6410 Spring 2015 Sample Midterm#1 Name: KEY**

**1**. Choosing from among **(REC)** **recursive**, **(RE)** **re non-recursive, (coRE) co-re non-recursive**, **(NRNC)** **non-re/non-co-re**, categorize each of the sets in a) through d). Justify your answer by showing some minimal quantification of some known recursive predicate.

**a.) { f | domain(f) is finite }** NRNC

**Justification: ∃x ∀y≥x ∀t ~STP(f, y, t)**

**b.) { f | domain(f) is empty }**  CO

**Justification: ∀x ∀t ~STP(f, x, t)**

**c.) { <f,x> | f(x) converges in at most 20 steps }**  REC

**Justification: STP(f, x, 20)**

**d.) { f | domain(f) converges in at most 20 steps for some input x }**  RE

**Justification: ∃x STP(f, x, 20)**

**2**. Let set **A** be recursive, **B** be re non-recursive and **C** be non-re. Choosing from among **(REC)** **recursive**, **(RE)** **re non-recursive**, **(NR)** **non-re**, categorize the set **D** in each of a) through d) by listing **all** possible categories. No justification is required.

**a.) D = ~C RE, NR**

**b.) D ⊆ A ∪ C REC, RE, NR**

**c.) D = ~B NR**

**d.) D = B − A REC, RE**

**3.** Prove that the **Halting Problem** (the set **HALT** **=** **K0 = Lu**) is not recursive (decidable) within any formal model of computation. (Hint: A diagonalization proof is required here.)

***Look at notes.***

**4.** Using reduction from the known undecidable **HasZero, HZ = { f | ∃x f(x) = 0 }**, show the non-recursiveness (undecidability) of the problem to decide if an arbitrary partial recursive function **g** has the property **IsZero, Z = { f | ∀x f(x) = 0 }**. Hint: there is a very simple construction that uses **STP** to do this. **Just giving that construction is not sufficient; you must also explain why it satisfies the desired properties of the reduction**.

***HZ = { f | ∃x ∃t [ STP(f, x, t) & VALUE(f, x, t) == 0] }***

***Let f be the index of an arbitrary effective procedure.***

***Define gf(y) = 1 - ∃x ∃t [ STP(f, x, t) & VALUE(f, x, t) == 0]***

***If ∃x f(x) = 0, we will find the x and the run-time t, and so we will return 0 (1 – 1)***

***If ∀x f(x) ≠ 0, then we will diverge in the search process and never return a value.***

***Thus, f ∈ HZ iff gf ∈ Z.***

**5.** Define **RANGE\_ALL = ( f | range(f) = ℵ }**.

**a.)** Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of this set. (Hint: Look at **c.)** and **d.)** to get a clue as to what this must be.)

**∀x ∃<y,t>[STP(f,y,t) && Value(f,y,t)=x]**

**b.)** Use Rice’s Theorem to prove that **RANGE\_ALL** is undecidable.

**This is non-trivial as I(x) = x ∈ RANGE\_ALL and C0(x) = 0 ∉ RANGE\_ALL**

**Let f,g be such that ∀x ϕf(x) = ϕg(x).**

**f∈ RANGE\_ALL ⇔ range(f) = ℵ**

**⇔ range(g) = ℵ since g outputs the same value as f for any input**

**⇔ g ∈ RANGE\_ALL**

**Since the property is non-trivial and is an I/O property, Rice’s Theorem says it is undecidable.**

**c.)** Show that **TOTAL ≤m RANGE\_ALL**, where **TOTAL = { f | ∀y ϕf(y)↓ }**.

**Let f be the index of an arbitrary effective procedure ϕf. Define g such that g(f), denoted gf, is the index of the function ϕgf defined by ϕgf(x) = ϕf(x) - ϕf(x)+x.**

**f ∈ TOTAL ⇔ ∀x ϕf(x)↓ ⇔ ∀x ϕgf(x) = x ⇒ ∀x x∈range(gf) ⇒ gf ∈ RANGE\_ALL**

**f ∉ TOTAL ⇔ ∃x ϕf(x)↑ ⇔ ∃x ϕgf(x)↑ ⇒ ∃x x∉range(gf) ⇒ gf ∉ RANGE\_ALL**

**This shows that TOTAL ≤m RANGE\_ALL, as was desired.**

**d.)** Show that **RANGE\_ALL ≤m TOTAL**.

**Let f be the index of an arbitrary effective procedure ϕf. Define g such that g(f), denoted gf, is the index of the function ϕgf defined by ϕgf(x) = ∃<y,t> [STP(f,y,t) & Value(f,y,t)=x].**

**f ∈ RANGE\_ALL ⇔ ∀x ∃<y,t> [STP(f,y,t) && Value(f,y,t)=x] ⇔ ∀x ϕgf(x)↓ ⇔ gf ∈ TOTAL**

**This shows that RANGE\_ALL ≤m TOTAL, as was desired.**

**e.)** From **a.)** through **d.)** what can you conclude about the complexity of **RANGE\_ALL**?

**a) shows that RANGE\_ALL is no more complex than others that must use the alternating qualifiers** **∀∃. b) shows the problem is non-recursive. c) and d) combine to show that the problem is in fact of equal complexity with the non-re problem TOTAL, so the result in a) was optimal.**

**6.** This is a simple question concerning Rice’s Theorem.

**a.)** State the strong form of Rice’s Theorem. Be sure to cover all conditions for it to apply.

**Let P be a property of indices of partial recursive function such that the set   
SP = { f | f has property P } has the following two restrictions**

1. **SP is non-trivial. This means that SP is neither empty nor is it the set of all indices.**
2. **P is an I/O behavior. That is, if f and g are two partial recursive functions whose I/O behaviors are indistinguishable, ∀x f(x)=g(x), then either both of f and g have property P or neither has property P.**

**Then P is undecidable.**

**b.)** Describe a set of partial recursive functions whose membership cannot be shown undecidable through Rice’s Theorem. What condition is violated by your example?

**There are many possibilities here. For example { f | ∃x ~STP(f,x,x) } is not an I/O property and   
{ f | ∃x f(x) ≠ f(x) } is trivial (empty).**

**7.** Using the definition that **S** is recursively enumerable iff **S** is either empty or the range of some algorithm **fS** (total recursive function), prove that if both **S** and its complement **~S** are recursively enumerable then **S** is decidable. To get full credit, you must show the characteristic function for **S**, **χS**, in all cases. Be careful to handle the extreme cases (there are two of them). Hint: This is not an empty suggestion.

**Let S = φ then ~S = ℵ. Both are re and ∀x χS(x) = 0 is S’s characteristic function.**

**Let S = ℵ then ~S = φ. Both are re and ∀x χS(x) = 1 is S’s characteristic function.**

**Assume then that S ≠ φ and S ≠ ℵ then each of S and ~S is enumerated by some total recursive function. Let S be enumerated by fS and ~S by f~S. Define**

**χS(x) = fS( μy [fS(y)==x || f~S(y)==x] ) == x.**

**Moreover, the minimization, while conceptually unbounded, always converges because both fS and by f~S are algorithms.**

**Further, x must be in the range of one and only one of fS or f~S. Thus,   
∃y fS (y) == x or ∃y f~S(y) == x.**

**The min operator (μy) finds the smallest such y and the predicate**

**fS( μy [fS(y)==x || f~S(y)==x] ) == x checks that x is in the range of fS.**

**If it is, then χS(x) = 1 else χS(x) = 0, as desired.**