The Master Theorem and Some Summation Formulas

A. Finite Summations.

Arithmetic Series: $\sum_{k=0}^{n-1}(a+kd) = a+(a+d)+\ldots+(a+(n-1)d) = \dfrac{n(2a+(n-1)d)}{2}$.

Geometric Series: $\sum_{k=0}^{n} x^k = 1+x+x^2+\ldots+x^n = \dfrac{1-x^{n+1}}{1-x}$, if $x \neq 1$.

The Binomial Theorem (finite summation case):

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}, \text{ where } \binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}.$$

$\sum_{k=0}^{n} kx^k = 0+x+2x^2+\ldots+nx^n = \dfrac{nx^{n+2}-(n+1)x^{n+1}+x}{(1-x)^2}$, if $x \neq 1$.

$\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$.

$\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$.

$\sum_{k=1}^{n} k^3 = \dfrac{n^2(n+1)^2}{4}$.

B. Infinite Series.

$\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, if $|x|<1$.

$\dfrac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$, if $|x|<1$.

$\dfrac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$, if $|ax|<1$.

$\dfrac{1}{(1-x)^n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k$, if $|x|<1$, where $n>0$ is a positive integer.

Two special cases (with $n=2$ and $n=3$) are:

$\dfrac{1}{(1-x)^2} = \sum_{k=0}^{\infty}\binom{k+1}{k}x^k = \sum_{k=0}^{\infty}(k+1)x^k$, and

$\dfrac{1}{(1-x)^3} = \sum_{k=0}^{\infty}\binom{k+2}{k}x^k = \sum_{k=0}^{\infty}\dfrac{(k+2)(k+1)}{2}x^k$.

The Binomial Theorem (general case):

$$(1+x)^r = \sum_{k=0}^{\infty}\binom{r}{k}x^k = 1+rx+\frac{r(r-1)}{2!}x^2+\frac{r(r-1)(r-2)}{3!}x^3+\ldots,$$

if $|x|<1$, where $\binom{r}{k} = \dfrac{r(r-1)\ldots(r-k+1)}{k!}$, r is a real number and $k \geq 0$ is an integer.

C. The Master Theorem (for recurrences):

Given a recurrence $T(n) = aT(n/b) + f(n)$, where n/b is interpreted as either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then the order of $T(n)$ can be determined as follows:

(1) if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \theta(n^{\log_b a})$.

(2) if $f(n) = \theta(n^{\log_b a})$, then $T(n) = \theta(n^{\log_b a} \cdot \lg n)$.

(3) if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af(n/b) \leq c\,f(n)$ for some constant $c < 1$ when n is sufficiently large, then $T(n) = \theta(f(n))$.