# COT5310: Formal Languages and Automata Theory

Lecture Notes #2: Decidability

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# Decidability

1. Introduction to decidability

2. Undecidability

3. Decidability

# 1. Introduction to decidability

When a formalism is developed, the following questions are crucial:

- expressive power – What can I say?
- decidable questions – What can I prove?
- complexity questions – How hard is to prove it?
- axiomatics – How should I prove it?

# 1. Introduction to decidability

What is an algorithmic problem? An algorithmic problem is a function $f : \mathcal{I} \to \mathcal{F}$, where $\mathcal{I}$ and $\mathcal{F}$ are two sets at most countable.

As we will only consider algorithmic problems, we will simply call them problems.

$\mathcal{I}$ is called the set of initial data or instances of $f$, and $\mathcal{F}$ is the set of final data.

When $|\mathcal{F}| = 2$ ($\mathcal{F} = \{0, 1\}$ or $\mathcal{F} = \{\top, \perp\}$ or $\mathcal{F} = \{yes, no\}$ etc.), $f$ is called a decision problem; otherwise, it is called a computational problem.

**Example 1**

- $f : \mathbf{N}^2 \to \mathbf{N}$ given by $f(x,y) = x + y$, is a computational problem (the "addition problem"). Each pair $(x,y) \in \mathbf{N}^2$ is an instance of this problem;
- $f : \mathbf{N} \to \{0,1\}$ given by $f(x) = 1$ if and only if $x$ is a prime, is a decision problem. Each $x \in \mathbf{N}$ is an instance of this problem.

Let $f : \mathcal{I} \rightarrow \mathcal{F}$ be a problem. As $\mathcal{I}$ and $\mathcal{F}$ are at most countable, they can be encoded as words over a given alphabet $\Sigma$. Therefore, we may assume that $\mathcal{I}, \mathcal{F} \subseteq \Sigma^*$.

**Example 2** Examples of encodings:

- $A \subseteq \mathbf{N} \quad \rightsquigarrow \quad \{a^x | x \in A\}$, over $\Sigma = \{a\}$;
- $A \subseteq \mathbf{N}^2 \quad \rightsquigarrow \quad \{a^x \# b^y | (x, y) \in A\}$, over $\Sigma = \{a, b, \#\}$;

# 1. Introduction to decidability

Turing machines are a good model for the study of algorithms, since we can conceive of

- computations with arbitrarily large inputs on their tapes, using an
- arbitrarily large amount of intermediate storage during a computation, and taking an
- arbitrarily large amount of time.

Moreover, Turing machines are universal, in the sense that every known algorithm can be executed by some Turing machine.

Consider the following algorithm:

```
Algorithm 𝒜
input:   x ∈ N;
output:  "yes", if x < 5, and "no", if x = 5;
begin
  i := x;
  while i > 5 do i := i + 1;
  if i < 5 then "yes" else if i = 5 then "'"no";
end.
```

- $accept(\mathcal{A}) = \{0, 1, 2, 3, 4\}$
- $reject(\mathcal{A}) = \{5\}$
- $loop(\mathcal{A}) = \{x \in \mathbf{N} | x > 5\}$

Let $f : \mathcal{I} \rightarrow \{0, 1\}$ be a decision problem, where $\mathcal{I} \subseteq \Sigma^*$. The language associated to $f$ is the set $L_f = \{w \in \mathcal{I} | f(w) = 1\}$.

$f$ is called decidable if its language is recursive.

> $f$ decidable $\Leftrightarrow$ there exists an algorithm (Turing machine) that decides $f$ ($L_f$)

$f$ is called semi-decidable if its language is recursively enumerable.

> $f$ semi-decidable $\Leftrightarrow$ there exists an algorithm (Turing machine) that semi-decides $f$ ($L_f$)

$f$ is called undecidable if it is not decidable.

A decision problem $f : \mathcal{I} \to \{0, 1\}$ is reducible to a decision problem $g : \mathcal{I}' \to \{0, 1\}$, abbreviated $f \prec g$, if there exists an algorithm (Turing machine) $M$ such that:

- $(\forall x \in \mathcal{I})(M(x) \in \mathcal{I}')$;
- $(\forall x \in \mathcal{I})(f(x) = 1 \quad \Leftrightarrow \quad g(M(x)) = 1)$.

**Proposition 1** Let $f$ and $g$ be decision problems.

- If $f \prec g$ and $g$ is decidable, then $f$ is decidable.
- If $f \prec g$ and $f$ is undecidable, then $g$ is undecidable.

# 2. Undecidability

# 2.1. The Halting Problem

2.1.1. The halting problem and its undecidability

2.1.2. Stack machines. Counter machines

2.1.3. Applications to Petri nets

2.1.4. Applications to security protocols

# 2.1.1. The Halting Problem and its Undecidability

The halting problem for a given algorithmic formalism is the problem of whether or not a given procedure of the formalism when executed with a given input eventually terminates.

The Halting Problem

Instance: algorithm $\mathcal{A}$ (Turing machine $M$) and input $x$;

Question: does $\mathcal{A}$ $(M)$ halt on $x$?

**Theorem 1** The halting problem for Turing machines is undecidable.

**Proof** Assume that there exists an algorithm $\mathcal{A}$ that decides the halting problem. Denote by $\langle\mathcal{B}\rangle$ an arbitrary but fixed encoding of an algorithm $\mathcal{B}$. Let $\mathcal{D}$ be the following algorithm:

<u>`Algorithm` $\mathcal{D}$</u>

**input**: algorithm $\mathcal{B}$;
**output**: 0 if $\mathcal{B}(\langle\mathcal{B}\rangle)\uparrow$;
**begin**
  $y := \mathcal{A}(\mathcal{B}, \langle\mathcal{B}\rangle)$;
  **if** $y = 0$ **then** 0 **else loop forever**
**end**.

It is easy to see that $\mathcal{D}(\langle\mathcal{D}\rangle)\uparrow \;\Leftrightarrow\; \mathcal{D}(\langle\mathcal{D}\rangle)\downarrow$, which is a contradiction. $\square$

# 2.1.1. The Halting Problem and its Undecidability

There are several variants on the halting problem.

The Empty-input Halting Problem
  Instance: algorithm $\mathcal{A}$ (Turing machine $M$);
  Question: does $\mathcal{A}$ ($M$) halt on the empty-input?

**Corollary 1** The empty-input halting problem for Turing machines is undecidable.

**Proof** We exhibit a reduction from the halting problem:

$$(\mathcal{A}, x) \quad \leadsto \quad \mathcal{A}'$$

where $\mathcal{A}'$, on the empty-input, generates $x$ and then simulates $\mathcal{A}$ on $x$.

Given an algorithm $\mathcal{A}$ and an input $x$ for it, define the algorithm $\mathcal{A}'$ as follows:

<u>**Algorithm** $\mathcal{A}'$</u>

**input**: none;

**output**: $z = \mathcal{A}(x)$;

**begin**

$z := \mathcal{A}(x)$;

**end**.

Clearly, $\mathcal{A}(x)\downarrow$ iff $\mathcal{A}'\downarrow$. $\square$

The Uniform Halting Problem

Instance: algorithm $\mathcal{A}$ (Turing machine $M$);

Question: does $\mathcal{A}$ ($M$) halt on all inputs?

**Corollary 2** The uniform halting problem for Turing machines is undecidable.

**Proof** We exhibit a reduction from the halting problem:

$$(\mathcal{A}, x) \quad \leadsto \quad \mathcal{A}'$$

where $\mathcal{A}'$, on an arbitrary input $y$, erases $y$, generates $x$, and then simulates $\mathcal{A}$ on $x$.

Given an algorithm $\mathcal{A}$ and an input $x$ for it, define the algorithm $\mathcal{A}'$ as follows:

<u>`Algorithm` $\mathcal{A}'$</u>

`input`: $y$;

`output`: $z = \mathcal{A}(x)$;

`begin`

  $z := \mathcal{A}(x)$;

`end`.

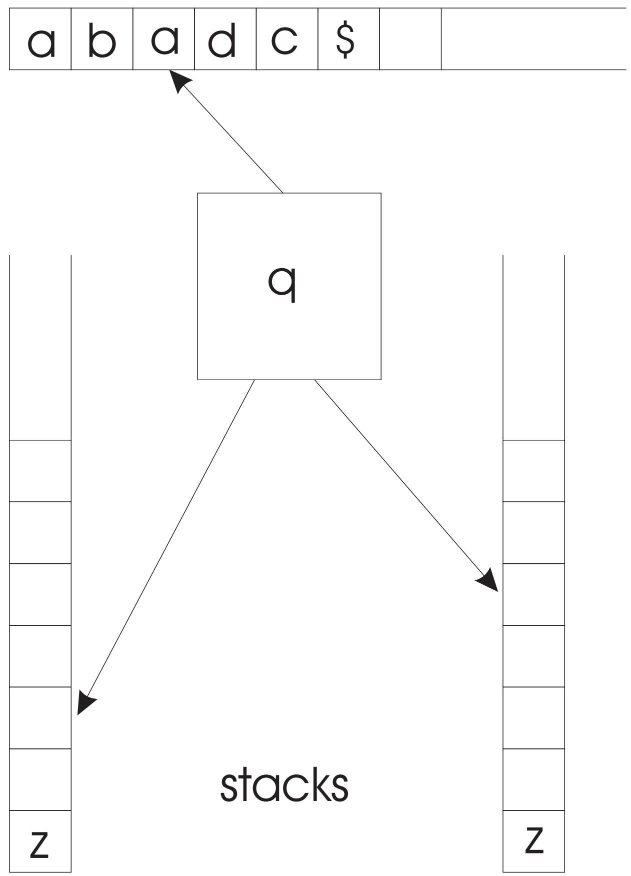
Clearly, $\mathcal{A}(x)\downarrow$ iff $(\forall y)(\mathcal{A}'(y)\downarrow$. $\Box$

# 2.1.2. Multistack Machines. Counter Machines

A $k$-stack machine, abbreviated k-SM, is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, Z, F)$, where:

- $Q$ is a non-empty finite set of states
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the final set of states
- $\Sigma$ is the input alphabet
- $\Gamma$ is the stack alphabet
- $Z \in \Gamma - \Sigma$ is the bottom-of-stack marker
- $\delta : Q \times (\Sigma \cup \{\lambda\}) \times (\Gamma)^k \leadsto Q \times (\Gamma^*)^k$ is the transition function satisfying the property that the bottom-of-stack marker $Z$ "cannot be erased" and it "cannot appear elsewhere on the stacks".

# 2.1.2. Multistack Machines. Counter Machines




input tape - read-only

$ - endmarker

Z - bottom-of-stack marker

Stack machine

Computation relation:

$$(q, u|av, Zu_1X_1, \ldots, Zu_kX_k) \vdash (q', ua|v, \gamma_1, \ldots, \gamma_k)$$

iff

$$\delta(q, a, X_1, \ldots, X_k) = (q', \gamma_1, \ldots, \gamma_k)$$

where $u, v \in \Sigma^*$, $a \in \Sigma \cup \{\lambda\}$, $u_1X_1, \ldots, u_kX_k \in \Gamma^*$.

**Theorem 2** A language is accepted by a Turing machine iff it is accepted by a 2-stack machine.

**Corollary 3** The halting problem for 2-stack machines is undecidable.

A k-counter machine, abbreviated k-CM, is a k-SM $M = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$ such that $|\Gamma| = 2$.

**Theorem 3** A language is accepted by a Turing machine iff it is accepted by a 2-CM.

**Corollary 4** The halting problem for 2-counter machines is undecidable.

## 2.1.2. Multistack Machines. Counter Machines

A "simplified' version of counter machines:

$$M = (Q, q_0, q_f, C, x_0, I),$$

where:

- $Q$ is a non-empty finite set of states;
- $q_0 \in Q$ is the initial state, and $q_f \in Q$ is the final state;
- $C$ is a finite set of counters, each of which being able to hold a natural number;
- $x_0 : C \rightarrow \mathbf{N}$ is the initial content of counters;
- $I$ is a finite set of instructions. For each state there is at most an instruction that can be executed at that state; for $q_f$ there is no instruction. Each instruction is of the one of the following forms:

– increment instruction $I(q, c, q')$

```
q :  begin
       c := c + 1;
       go to q'
     end
```

– test instruction $I(q, c, q', q'')$

```
q :  begin
       if c = 0 then go to q'
        else begin
                   c := c − 1;
                   go to q''
                 end
     end
```

A configuration is a pair $(q, x)$, where $q \in Q$ and $x : C \rightarrow N$. A configuration $(q, x)$ is called initial if $q = q_0$ and $x = x_0$. A configuration $(q, x)$ is called final if $q = q_f$.

Computation:

$$(q, x) \vdash (q', x')$$

iff one of the following holds:

- there exists $I(q, c, q')$ such that $x'(c) = x(c)+1$ and $x'(c') = x(c')$, for all $c' \in C - \{c\}$;
- there exists $I(q, c, q_1, q_2)$ such that
  - if $x(c) = 0$, then $q' = q_1$ and $x' = x$;
  - if $x(c) \neq 0$, then $q' = q_2$, $x'(c) = x(c) - 1$, and $x'(c') = x(c)$, for all $c' \in C - \{c\}$.

# 2.1.3. Applications to Petri Nets

Petri nets, abbreviated PN, have been introduced by Carl Adam Petri in 1962 as models of distributed systems, where concurrency and communication play an important role.

A PN is a system $\Sigma = (S, T, F, W)$, where:

- $S$ is a finite non-empty set of places;
- $T$ is a finite non-empty set of transitions;
- $S \cap T = \emptyset$;
- $F \subseteq S \times T \cup T \times S$ is the flow relation;
- $W : S \times T \cup T \times S \rightarrow \mathbf{N}$ is the weight function satisfying $W(x, y) = 0$ iff $(x, y) \notin F$.

## 2.1.3. Applications to Petri Nets

Configurations in Petri net theory are called markings, and they are defined as functions $M : S \rightarrow \mathbf{N}$.

Because $S$ is a finite set, markings are usually represented as $S$-dimensional vectors.

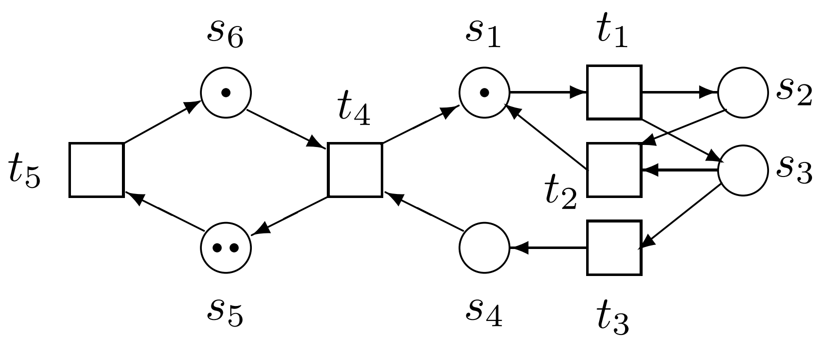
Computation (firing) rule:

- A transition $t$ is enabled at $M$, denoted $M[t\rangle$, if
$$W(s,t) \geq M(s),$$
for all $s \in S$;
- If $t$ is enabled at $M$ then $t$ may fire yielding a new marking $M'$ given by
$$M'(s) = M(s) - W(s,t) + W(t,s),$$
for all $s \in S$. We denote this by $M[t\rangle M'$.

# 2.1.3. Applications to Petri Nets

Graphical representation:

**Example 3** Vending machine:




$s_1$ = *ready*
$s_2$ = *counter*
$s_3$ = *inserted*
$s_4$ = *accepted*
$s_5$ = *warm*
$s_6$ = *cold*

$t_1$ = *insert*
$t_2$ = *reject*
$t_3$ = *accept*
$t_4$ = *dispense*
$t_5$ = *brew*

$$(1,0,0,0,2,1)[t_1\rangle(0,1,1,0,2,1)[t_3\rangle(0,1,0,1,2,1)[t_4\rangle(1,1,0,0,3,0)$$

# 2.1.3. Applications to Petri Nets

A pair $\gamma = (\Sigma, M_0)$, where $\Sigma$ is a Petri net and $M_0$ is a marking of $\Sigma$ is called a marked Petri net.

A marking $M$ is reachable in $\gamma$ if there exists a sequence of transitions $w \in T^*$ such that $M_0[w\rangle M$.

A marking $M$ is coverable in $\gamma$ if there exists a reachable marking $M'$ in $\gamma$ such that $M' \geq M$ (the inequality on vectors is componentwise understood).

$\gamma$ is bounded if there exists $n \in \mathbf{N}$ such that $M(s) \leq n$, for any $s \in S$ and reachable marking $M$.

A transition $t$ of $\gamma$ is live if for any reachable marking $M$ there exists $M'$ reachable from $M$ such that $M'[t\rangle$. If all transitions are live, the $\gamma$ is called live.

# 2.1.3. Applications to Petri Nets

Basic decision problems in Petri net theory: reachability, coverability, boundedness, and liveness.

All these problems are decidable for Petri nets (details will be provided in a separate section). However, they are undecidable for almost all Petri net extensions. For instance, we will prove that they are undecidable for inhibitor Petri nets (see `InhibitorPetriNets.pdf`).

# 2.1.4. Applications to Security Protocols

see `SecurityProtocols.pdf`

# 2.3. Post's Correspondence Problem

# 2.3.1. Post's Correspondence Problem

- Emil Post. *A Variant of a Recursively Unsolvable Problem*, Bulletin of the AMS 52, 1946, 264–268 (see `Post1946.pdf`).

Post's Correspondence Problem (PCP)

| | |
|---|---|
| Instance: | list of pairs of words $L = \{(u_1, v_1), \ldots, (u_n, v_n)\}$ |
| Question: | Is there any list of numbers $i_1, \ldots, i_k$ s.t. $u_{i_1} \cdots u_{i_k} = v_{i_1} \cdots v_{i_k}$? |

Any list of numbers $i_1, \ldots, i_k$ such that

$$u_{i_1} \cdots u_{i_k} = v_{i_1} \cdots v_{i_k}$$

is called a solution of $L$.

**Example 4** The list 1,2,1,3 is a solution to the PCP instance

$$L = \{(a^2, a^2b), (b^2, ba), (ab^2, b)\}$$

**Example 5** The PCP instance

$$L = \{(a^2b, a^2), (a, ba^2)\}$$

has no solution.

**Example 6** The list 1,3,2,3 is a solution to the PCP instance

$$L = \{(1, 101), (10, 00), (011, 11)\}$$

**Proposition 2** The PCP instance

$$L = \{(a^{k_1}, a^{l_1}), \ldots, (a^{k_n}, a^{l_n})$$

has solutions if and only if

1. there exists $i$ such that $k_i = l_i$, or
2. there exist $i$ and $j$ such that $k_i > l_i$ and $k_j < l_j$.

**Corollary 5** PCP over one-letter alphabets is decidable.

**Proposition 3** Any PCP instance over an alphabet $\Sigma$ with $|\Sigma| \geq 2$ is equivalent to a PCP instance over an alphabet $\Delta$ with $|\Delta| = 2$.

**Proof** Assume $\Sigma = \{a_1, \ldots, a_n\}$ and $n > 2$. Let $\Delta = \{a, b\}$, where $a \neq b$.

Encode $a_i$ by $ba^ib$, for any $i$. $\square$

Define:

- $PCP_1$ – PCP instances over one-letter alphabets
- $PCP_2$ – PCP instances over two-letter alphabets

**Theorem 4** PCP is undecidable.

**Proof** Show that the halting problem for Post machines, which are equivalent to Turing machines, can be reduced to PCP. □

**Corollary 6** $PCP_2$ is undecidable.

Summary:

- $PCP_1$ is decidable
- $PCP_2$ is undecidable

Other variations:

- $PCP(n)$ – PCP instances of length $n$

$$(L = \{(u_1, v_1), \ldots, (u_n, v_n)\})$$

- $PCP_1(n)$
- $PCP_2(n)$

$PCP_1(n)$ is decidable, for all $n$.

**Theorem 5** (Ehrenfeucht, Karhumaki, Rozenberg, 1982)
PCP(2) is decidable.

For a simpler proof than the original one see `HaHH2000.pdf`.

**Theorem 6** Matiyasevich, Senizergues, 1996)
PCP(7) is undecidable.

**Proof** See `MaSe1996.pdf`. □

Open problems: PCP(3),...,PCP(6)

# 2.3.2. Applications of PCP to First-order Logic

Validity problem for first-order logic (VPFOL)

Instance: First-order formula $\phi$

Question: Is $\phi$ valid?

Satisfiability problem for first-order logic (SPFOL)

Instance: First-order formula $\phi$

Question: Is $\phi$ satisfiable?

These two decision problems are equivalent because

$$\phi \text{ is valid} \quad \Leftrightarrow \quad \neg\phi \text{ is not satisfiable}$$

**Theorem 7** VPFOL is undecidable.

**Proof** Reduce $PCP_2$ to VPFOL. Given a $PCP_2$ instance

$$L = \{(u_1, v_1), \ldots, (u_n, v_n)\}$$

over $\Sigma = \{0, 1\}$, define a formula $\phi$ such that

$$L \text{ has solutions} \quad \Leftrightarrow \quad \phi \text{ is valid}$$

$\phi$ is defined as follows:

- let $a$ be a constant. It will interpreted by $\lambda$ in some interpretation $\mathcal{I}$;
- let $f_0$ and $f_1$ be function symbols. They will be interpreted by $\mathcal{I}(f_0)(x) = x0$ and $\mathcal{I}(f_1)(x) = x1$. We will simply write $f_{b_1 \cdots b_k}(x)$ instead of $f_{b_k}(\cdots f_{b_1}(x) \cdots)$;

# 2.3.2. Applications of PCP to First-order Logic

- let $P$ be a predicate symbol. It will be interpreted by

$$\mathcal{I}(P)(x,y) \;\Leftrightarrow\; x,y \in \Sigma^* \wedge x = u_{i_1} \cdots u_{i_k} \wedge y = v_{i_1} \cdots v_{i_k},$$

$$\text{for some } i_1, \ldots, i_k$$

- let $\phi_1$ be the formula $\phi_1 = \wedge_{i=1}^{k} P(f_{u_i}(a), f_{v_i}(a))$
- let $\phi_2$ be the formula $\phi_2 = (\forall u,v)(P(u,v) \Rightarrow \wedge_{i=1}^{k} P(f_{u_i}(u), f_{v_i}(u))$
- let $\phi_3$ be the formula $\phi_3 = (\exists x)(P(x,x))$
- let $\phi$ be the formula $\phi = (\phi_1 \;\wedge\; \phi_2 \Rightarrow \phi_3)$

Then,

$$L \text{ has solutions } \;\Leftrightarrow\; \phi \text{ is valid}$$

which concludes the proof. □

Intersection problem for CFL (IPCFL)

Instance: context-free grammars $G_1$ and $G_2$

Question: Is $L(G_1) \cap L(G_2) \neq \emptyset$?

**Theorem 8** IPCFL is undecidable.

**Proof** Reduce PCP to IPCFL. Given a PCP instance

$$L = \{(u_1, v_1), \ldots, (u_n, v_n)\}$$

over $\Sigma$, define two CF-grammars $G_1$ and $G_2$ such that

$$L \text{ has solutions } \Leftrightarrow L(G_1) \cap L(G_2) \neq \emptyset$$

The grammars are:

- $G_1$ : $S \rightarrow iSu_i | iu_i$, for all $i$
- $G_2$ : $S \rightarrow iSv_i | iv_i$, for all $i$

□

# 2.3.3. Applications of PCP to Formal Language Theory

Equivalence problem for CFG (EPCFG)

Instance: context-free grammars $G_1$ and $G_2$

Question: Is $L(G_1) = L(G_2)$?

**Theorem 9** EPCFG is undecidable.

**Proof** Reduce $\neg$PCP to EPCFG. Given a PCP instance

$$L = \{(u_1, v_1), \ldots, (u_n, v_n)\}$$

over $\Sigma$, define two CF-grammars $G_1$ and $G_2$ such that

$$L \text{ has no solution } \Leftrightarrow L(G_1) = L(G_2)$$

Define two grammars $G_1$ and $G_2$ such that

- $G_1$ generates $L(G_1) = \{1, \ldots, n\}^* \# \Sigma^*$, where $\#$ is a new symbol;

## 2.3.3. Applications of PCP to Formal Language Theory

- $G_2$ generates $L(G_2) = (L(G_1) - A) \cup (L(G_1) - B)$, where
  - $A = \{i_1 \cdots i_k \# u_{i_k} \cdots u_{i_1} | i_1, \ldots, i_k \in \{1, \ldots, n\}\}$
  - $B = \{i_1 \cdots i_k \# v_{i_k} \cdots v_{i_1} | i_1, \ldots, i_k \in \{1, \ldots, n\}\}$

It is easy to see that two context-free grammars $G_1$ and $G_2$ as above exist, and $L$ has no solution iff $L(G_1) = L(G_2)$. $\square$

Ambiguity problem for CFG (APCFG)

Instance: context-free grammar $G$
Question: Is $G$ ambiguous?

**Theorem 10** APCFG is undecidable.

**Proof** Reduce PCP to APCFG. Given a PCP instance

$$L = \{(u_1, v_1), \ldots, (u_n, v_n)\}$$

over $\Sigma$, define a CFG $G$ such that

$$L \text{ has solutions} \quad \Leftrightarrow \quad G \text{ is ambiguous}$$

Define $G$ by

- $S \rightarrow S_1 | S_2$, $\quad S_1 \rightarrow u_i S_1 i | u_i i$, $\quad S_2 \rightarrow v_i S_2 i | v_i i$,

for all $i$. □

A semi-group $(S, \cdot)$ is called finitely presented if there exists a finite set $A$ of generators for $S$ and a finite set $E$ of equations over $A$ (i.e., pairs of words over $A$) such that any valid equation in $S$ can be obtained by derivation from $E$. That is, if $t = t'$ is valid in $S$, then $t \overset{*}{\Rightarrow}_E t'$.

**Example 7** Let $S$ be a semi-group generated by $A = \{a_1, a_2, a_3\}$ under the equations

- $a_2a_1 = a_1a_2$
- $a_3a_2 = a_2a_2a_3$
- $a_3a_1 = a_1$.

Then, $a_1a_2a_2 = a_1a_2$ is valid in $S$.

# 2.6. The word problem for finitely presented monoids

<u>Word Problem for Semi-groups (WPSG)</u>

| | |
|---|---|
| Instance: | finite semi-group presentation $(A, E)$ and equation $t = t'$ |
| Question: | Does $t = t'$ hold true in the semi-group presented by $(A, E)$? |

This problem was shown to be undecidable in:

- Emil Post. *Recursive Unsolvability of a Problem of Thue*, Journal of Symbolic Logic 12, 1947, 1–11.

The problem can be reduced to the reachability problem for Thue systems.

- Axel Thue. *Probleme über Veranderungen von Zeichenreihen nach gegeben regeln*, Skr. Vid. Kristiania, I Mat. Naturv. Klasse 10, 1914.

A Thue system over an alphabet $\Sigma$ is any set of unordered pairs of words over $\Sigma$. Each pair $\{t, t'\}$ is usually written as $t = t'$.

A semi-Thue system over an alphabet $\Sigma$ is any set of ordered pairs of words over $\Sigma$. Each pair $(t, t')$ is usually written as $t \rightarrow t'$.

Reachability Problem for Semi-Thue Systems (RPSTS)

| | |
|---|---|
| Instance: | semi-Thue system $R$ and words $t$ and $t'$ |
| Question: | Does $t \stackrel{*}{\Rightarrow}_R t'$? |

**Theorem 11** The reachability problem for (semi-)Thue systems is undecidable.

**Proof** Reduce the halting problem for Turing machines to this problem. □

**Corollary 7** The word problem for finitely presented semigroups (monoids) is undecidable.

# 2.6. The word problem for finitely presented monoids

Term rewriting systems and related problems:

TermRewritingSystems.pdf

# 3. Decidability

Techniques for proving decidability:

- reducibility: if a problem $A$ is reducible to a problem $B$ and $B$ is decidable, then $A$ is decidable;
- ad hoc techniques.

## Coverability tree based techniques

General remarks:

- a coverability tree reduces the analysis of an infinite state space to the analysis of a finite state space;
- cut off infinite branches and add extra information to the leaf nodes;
- some properties of the original state space (reachability tree) may be lost.

We illustrate the technique on vector addition systems.

A vector addition system (VAS) is a couple $\mathcal{W} = (W, v_0)$, where:

- $W$ is a finite set of $n$-dimensional vectors with integer components ($W = \{v_1, \ldots, v_k\} \subseteq \mathbf{Z}^n$);
- $v_0$ is an $n$-dimensional vector with positive integer components ($v_0 \in \mathbf{N}^n$).

**Example 8** $\mathcal{W} = (W, v_0)$, where

$$W = \{(-1,1,0,1),(0,-1,0,0),(1,0,0,-1),(0,0,-1,1)\}$$

and $v_0 = (1,0,0,1)$, is a vector addition system.

Let $\mathcal{W} = (W, v_0)$ be a VAS, $x \in \mathbf{Z}^n$, and $v \in W$.

- $v$ is enabled at $x$, denoted $x[v\rangle$ or $x \xrightarrow{v}$, if $x + v \geq 0$. $W(x)$ stands for the set $\{v \in W | x[v\rangle\}$;
- if $v$ is enabled at $x$ then $v$ may be applied yielding a new vector $x'$ given by $x' = x + v$. We denote this by $x[v\rangle x'$ or $x \xrightarrow{v} x'$;
- $\Rightarrow = \bigcup_{v \in W} \xrightarrow{v}$;
- $\overset{+}{\Rightarrow}$ is the reflexive and transitive closure of $\Rightarrow$;
- $x$ is reachable in $\mathcal{W}$ if $d \overset{+}{\Rightarrow} x$;
- $[v_0\rangle$ is the set of all reachable vectors in $\mathcal{W}$, called the reachability set of $\mathcal{W}$;
- $x$ is coverable in $\mathcal{W}$ if $v_0 \overset{+}{\Rightarrow} x'$ and $x' \geq x$, for some $x'$;
- $v$ is dead in $\mathcal{W}$ if $\neg(x[v\rangle)$, for any $x$ reachable in $\mathcal{W}$.

# 3. Decidability

Let $\mathcal{W} = (W, v_0)$ be a VAS. A labeled tree $\mathcal{R} = (V, E, l_1, l_2)$ is a reachability tree of $\mathcal{W}$ if:

1. its root $x_0$ is labeled by $v_0$, i.e., $l_1(x_0) = v_0$;
2. $\forall x \in V$, $|x^+| = |W(l_1(x))|$;
3. $\forall x \in V$ with $|x^+| > 0$ and $\forall v \in W(l_1(x))$ there exists $x' \in x^+$ such that:
   (a) $l_1(x') = l_1(x) + v$;
   (b) $l_2(x, x') = v$.

Any two reachability trees of $\mathcal{W}$ are isomorphic. Therefore, we may talk about the reachability tree of $\mathcal{W}$, denoted $\mathcal{R}(\mathcal{W})$.

**Proposition 4** Let $\mathcal{W} = (W, v_0)$ be a VAS. Then,

1. $\mathcal{R}(\mathcal{W})$ is finitely branched;
2. $x$ in $\mathcal{R}(\mathcal{W})$ is a leaf node iff no vector in $W$ is enabled at $l_1(x)$;
3. $[v_0\rangle = \{l_1(x) | x \in V\}$.

$\mathcal{R}(\mathcal{W})$ may be infinite even if $[v_0\rangle$ is finite !

We will derive a finite structure from $\mathcal{R}(\mathcal{W})$.

Let $\omega \notin \mathbf{Z}$ and $\mathbf{Z}_\omega = \mathbf{Z} \cup \{\omega\}$. Extend $+$ and $<$ to $\mathbf{Z}_\omega$ by:

- $n + \omega = \omega + n = \omega$, for any $n \in \mathbf{Z}$;
- $n < \omega$, for any $n \in \mathbf{Z}$.

The notation $x[v\rangle$ etc. is usually extended to vectors over $\mathbf{Z}_\omega$.

Let $\mathcal{W} = (W, v_0)$ be a VAS. A labelled tree $\mathcal{T} = (V, E, l_1, l_2)$ is a coverability tree of $\mathcal{W}$ if:

1. its root $x_0$ is labelled by $v_0$, i.e., $l_1(x_0) = v_0$;
2. $\forall x \in V$,
$$|x^+| = \begin{cases} 0, & W(l_1(x)) = \emptyset \ or \\ & (\exists x' \in d_{\mathcal{T}}(x_0, x))(x \neq x' \ \wedge \ l_1(x) = l_1(x')) \\ |W(l_1(x))|, & \text{otherwise} \end{cases}$$
3. $\forall x \in V$ with $|x^+| > 0$ and $\forall v \in W(l_1(x))$ there exists $x' \in x^+$ such that:
   (a) $l_1(x')(i) = \omega$ if $(\exists x'' \in d_{\mathcal{T}}(x_0, x))(l_1(x'') \leq l_1(x) + v \ \wedge \ l_1(x'')(i) < (l_1(x)+v)(i))$, and $l_1(x')(i) = (l_1(x)+v)(i)$, otherwise (for any $i$);
   (b) $l_2(x, x') = v$.

Any two coverability trees of $\mathcal{W}$ are isomorphic. Therefore, we may talk about the coverability tree of $\mathcal{W}$, denoted $\mathcal{T}(\mathcal{W})$.

**Proposition 5** Let $\mathcal{W} = (W, v_0)$ be a VAS and $\mathcal{T}(\mathcal{W}) = (V, E, l_1, l_2)$ its coverability tree. Then:

1. $\mathcal{T}(\mathcal{W})$ is finitely branched;
2. $x$ in $\mathcal{T}(\mathcal{W})$ is a leaf node iff $W(l_1(x)) = \emptyset$ or there exists $x' \in d_{\mathcal{T}}(x_0, x)$ such that $x \neq x'$ and $l_1(x) = l_1(x')$;
3. let $x_{i_0}, x_{i_1}, \ldots, x_{i_m}$ be pairwise distinct nodes such that $x_{i_j} \in d_{\mathcal{T}(\gamma)}(x_0, x_{i_{j+1}})$, for any $0 \leq j \leq m-1$.
   (a) if $l_1(x_{i_0}) = l_1(x_{i_1}) = \cdots = l_1(x_{i_m})$, then $m \leq 1$;
   (b) if $l_1(x_{i_0}) < l_1(x_{i_1}) < \cdots < l_1(x_{i_m})$, then $m \leq n$;
4. $\mathcal{T}(\gamma)$ is finite.

**Theorem 12** Let $\mathcal{W} = (W, v_0)$ be a VAS and $\mathcal{T}(\mathcal{W}) = (V, E, l_1, l_2)$ its coverability tree. Then, a vector $x$ is coverable in $\mathcal{W}$ iff it is coverable in $\mathcal{T}(\mathcal{W})$.

**Corollary 8** Coverability, deadness and finiteness problems are decidable for VASs.