COT 5310 Fall 2007 Midterm#2 Name: KEY

12 **1**. Choosing from among **(REC) recursive**, **(RE) re non-recursive**, **(CO) co-re non-recursive**, **(NR) non-re**, categorize each of the sets in a) through d). Justify your answer by showing some minimal quantification of some known recursive predicate.

a.) $\{ \langle f,x \rangle \mid f(x)$ **takes at least** x^2 **steps to converge** $\}$ **REC**

Justification: $\sim STP(x,f,x^2-1)$

b.) $\{ f \mid range(f)$ **contains only even numbers** $\}$ **CO**

Justification: $\forall \langle x,t \rangle [STP(x,f,t) \Rightarrow isEven(x)]$

c.) $\{ f \mid range(f)$ **is not the set of natural numbers** $\}$ **NR**

Justification: $\exists x\ \forall \langle y,t \rangle [STP(y,f,t) \Rightarrow Value(y,f,t) \neq x]$

d.) $\{ f \mid f$ **converges on some pair of input, x, 2x** $\}$ **RE**

Justification: $\exists \langle x,t \rangle\ [STP(x,f,t)\ \&\&\ STP(2x,f,t)]$

9 **2**. Let **A** be re, possibly recursive, and **B** be re non-recursive. Let $\mathbf{C = (A \cap \sim B) \cup (B \cap \sim A)}$.
For each part, either show sets **A** and **B** with the specified property and **justify in detail** how these meet the required property, or present a **demonstration** that this property cannot hold.

a.) Can **C** be re non-recursive?

YES. Let A = ϕ. A is clearly re, even recursive since it is trivially decided by $\chi_A(x) = 0$. Then C = $(\phi \cap \sim B) \cup (B \cap \aleph)$ = B. B is given to be re, non-recursive.

b.) Can **C** be co-re non-recursive?

YES. Let A = ℵ. A is clearly re, even recursive since it is trivially decided by $\chi_A(x) = 1$. Then C = $(\aleph \cap \sim B) \cup (B \cap \phi)$ = ~B. Since B is given to be re, non-recursive, its complement must be co-re non-recursive, as desired.

12 **3**. Let set **A** and **B** be sets, such that $\mathbf{A \leq_m B}$ by the total m-1 recursive function $\mathbf{f_{AB}}$. For each of the following, be **complete** by addressing whether or not the specified set can be recursive, re non-recursive and/or non-re.

a.) Assume **A** is re, non-recursive and semi-decided by the partial recursive functions $\mathbf{g_A}$. What can we say about the complexity (recursive, re, non-re) of **B**? Address all three cases.

B is definitely not recursive and may not even be re.

B cannot be Recursive: Assume otherwise, and let B be decided by the characteristic function χ_B, then A would be decided by the characteristic function $\chi_B \circ f_{AB}$. That is, $x \in A$ iff $\chi_B(f_{AB}(x))$.

Since A is non-recursive, this is a contraction, and hence B cannot be recursive.

B can be RE, non-recursive: Let A=B then A≤mB using the reduction $f_{AB}(x) = x$ since $x \in A$ iff $f_{AB}(x) = x \in B$, which is precisely what we want since A=B.

B can be non-RE: Choose B = $\{ 2f \mid f \in TOTAL\} \cup \{2f+1 \mid f \in A\}$. Letting $f_{AB}(x) = 2x+1$, we can see that A≤mB. However, B has at least the complexity of TOTAL, since TOTAL≤mB by the mapping $x \in TOTAL$ iff $2x \in B$. Since TOTAL is non-RE, we have the desired result.

b.) Assume **B** is re, non-recursive and semi-decided by the partial recursive functions $\mathbf{g_B}$. What can we say about the complexity (recursive, re, non-re) of **A**? Address all three cases.

A is re and possibly recursive.

A can be Recursive: Let A=ℵ; $\chi_A(x) = 1$. Let $b \in B$ (there is some such b since B cannot be empty, else it would be recursive), then A≤mB using the reduction $f_{AB}(x) = b$ since $x \in A$ iff $f_{AB}(x) = b \in B$, which is true for all x and what we desire since A=ℵ.

A can be RE, non-recursive: Let A=B then A≤mB using the reduction $f_{AB}(x) = x$ since $x \in A$ iff $f_{AB}(x) = x \in B$, which is precisely what we want since A=B.

A cannot be non-RE: $x \in A$ iff $g_B(f_{AB}(x))\downarrow$, and thus is semi-decided by $g_A(x) = g_B(f_{AB}(x))$.

4. Define **RANGE_ALL = (f | range(f) = ℵ }**.

2 **a.)** Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of this set. (Hint: Look at **c.)** and **d.)** to get a clue as to what this must be.)

∀x ∃<y,t>[STP(y,f,t) & Value(y,f,t)=x]

5 **b.)** Use Rice's Theorem to prove that **RANGE_ALL** is undecidable.
This is non-trivial as I(x) = x ∈ RANGE_ALL and C_0(x) = 0 ∉ RANGE_ALL
Let f,g be such that ∀x $\varphi_f(x) = \varphi_g(x)$.
f∈ RANGE_ALL⇔ range(f) = ℵ
⇔ range(g) = ℵ since g outputs the same value as f for any input
⇔ g ∈ RANGE_ALL
Since the property is non-trivial and is an I/O property, Rice's Theorem says it is undecidable.

5 **c.)** Show that **TOTAL $\leq_m$ RANGE_ALL**, where **TOTAL = { f | ∀y $\varphi_f(y)\downarrow$ }**.

Let f be the index of an arbitrary effective procedure φ_f. Define g such that g(f), denoted g_f, is the index of the function φ_{g_f} defined by ∀x $\varphi_{g_f}(x) = \varphi_f(x) - \varphi_f(x) + x$.

f ∈ TOTAL ⇔ ∀x $\varphi_f(x)\downarrow$ ⇔ ∀x $\varphi_{g_f}(x) = x$ ⇒ ∀x x∈range(g_f) ⇒ g_f ∈ RANGE_ALL

f ∉ TOTAL ⇔ ∃x $\varphi_f(x)\uparrow$ ⇔ ∃x $\varphi_{g_f}(x)\uparrow$ ⇒ ∃x x∉range(g_f) ⇒ g_f ∉ RANGE_ALL

This shows that TOTAL $\leq_m$ RANGE_ALL, as was desired.

5 **d.)** Show that **RANGE_ALL $\leq_m$ TOTAL**.

Let f be the index of an arbitrary effective procedure φ_f. Define g such that g(f), denoted g_f, is the index of the function φ_{g_f} defined by ∀x $\varphi_{g_f}(x)$ = ∃<y,t> [STP(y,f,t) & VALUE(y,f,t)==x)]

This is ∀x $\varphi_{g_f}(x)$ = ∃y $\varphi_f(y)$==x, but it avoids the potential problem that $\varphi_f(y')\uparrow$, y'<y.

f ∈ RANGE_ALL ⇔ ∀x ∃y $\varphi_f(y)$=x ⇔ ∀x $\varphi_{g_f}(x)\downarrow$ ⇔ g_f ∈ TOTAL

This shows that RANGE_ALL $\leq_m$ TOTAL, as was desired.

2 **e.)** From **a.)** through **d.)** what can you conclude about the complexity of **RANGE_ALL**?
a) shows that RANGE_ALL is no more complex than others that must use the alternating qualifiers ∀∃. b) shows the problem is non-recursive. c) and d) combine to show that the problem is in fact of equal complexity with the non-re problem TOTAL, so the result in a) was optimal.

5. This is a simple question concerning Rice's Theorem.

4 **a.)** State the strong form of Rice's Theorem. Be sure to cover all conditions for it to apply.
Let P be a property of indices of partial recursive function such that the set
S_P = { f | f has property P } has the following two restrictions
(1) S_P is non-trivial. This means that SP is neither empty nor is it the set of all indices.
(2) P is an I/O behavior. That is, if f and g are two partial recursive functions whose I/O behaviors are indistinguishable, ∀x f(x)=g(x), then either both of f and g have property P or neither has property P.
Then P is undecidable.

2 **b.)** Describe a set of partial recursive functions whose membership cannot be shown undecidable through Rice's Theorem. What condition is violated by your example?
There are many possibilities here. For example { f | ∃x ~STP(x,f,x) } is not an I/O property and { f | ∃x f(x) ≠ f(x) } is trivial (empty).

8 **6.** Using the definition that **S** is recursively enumerable iff **S** is either empty or the range of some algorithm $\mathbf{f_S}$ (total recursive function), prove that if both **S** and its complement **~S** are recursively enumerable then **S** is decidable. To get full credit, you must show the characteristic function for **S**, χ_S, in all cases. Be careful to handle the extreme cases (there are two of them). Hint: This is not an empty suggestion.

Let S = ϕ then ~S = $\aleph$. Both are re and $\forall x\ \chi_S(x) = 0$ is S's characteristic function.

Let S = $\aleph$ then ~S = ϕ. Both are re and $\forall x\ \chi_S(x) = 1$ is S's characteristic function.

Assume then that S ≠ ϕ and S ≠ $\aleph$ then each of S and ~S is enumerated by some total recursive function. Let S be enumerated by f_S and ~S by $f_{\sim S}$. Define

$\chi_S(x) = f_S(\ \mu y\ [f_S(y)==x \ ||\ f_{\sim S}(y)==x]\ ) == x$.

Note first that f_S and $f_{\sim S}$ are total and so the above is well-defined.
Note also that x must be in the range of one and only one of f_S or $f_{\sim S}$. Thus,
$\exists y\ f_S\ (y) == x$ or $\exists y\ f_{\sim S}(y) == x$.

The min operator (μy) finds the smallest such y and the predicate

$f_S(\ \mu y\ [f_S(y)==x \ ||\ f_{\sim S}(y)==x]\ ) == x$ checks that x is in the range of f_S.

If it is, then $\chi_S(x) = 1$ else $\chi_S(x) = 0$, as desired.