1. Using reduction from the complement of the Halting Problem, show the undecidability of the problem to determine if an arbitrary partial recursive function, \( f \), has a summation upper bound. This means that there is an \( M \), such that the sum of all values in the range of \( f \) (repeats are added in and divergence just adds 0) is \( \leq M \).

The set \( \text{HALT} = \{ \langle f, x \rangle : f(x) \downarrow \} \), therefore \( \text{CoHALT} = \{ \langle f, x \rangle : f(x) \uparrow \} \). The set of partial functions \( f \) with a summation upper bound can be described as

\[
\text{UB} = \{ f : \exists x \forall y \left(y > x \Rightarrow (f(y) = 0 \text{ or } f(y) \uparrow)\right) \}.
\]

or in other words, only finitely many of the outputs can be non-zero.

To show that \( \text{CoHALT} \leq_m \text{UB} \) we define \( g(\langle f, x \rangle) = g_{f,x} \) as

\[
g_{f,x}(y) = \mu t \left[ \text{STP}(f, x, t) \right] + 1.
\]

Then if \( \langle f, x \rangle \in \text{CoHALT}, g_{f,x}(y) \uparrow \) for all \( y \in \mathbb{N} \) so \( \text{dom}(g_{f,x}) = \emptyset \Rightarrow g_{f,x} \in \text{UB} \).

If \( \langle f, x \rangle \notin \text{CoHALT}, g_{f,x}(y) \geq 1 \) for all \( y \in \mathbb{N} \). This means that no summation upper bound exists and \( g_{f,x} \notin \text{UB} \).

2. Use one of the versions of Rice’s Theorem to show the undecidability of the problem to determine if an arbitrary partial recursive function, \( f \), has a summation upper bound. This means that there is an \( M \), such that the sum of all values in the range of \( f \) (repeats are added in and divergence just adds 0) is \( \leq M \).

We can consider using Rice’s Theorem because UB is a set of partial function indices.

UB is non-trivial because if \( f(x) = 0 \) and \( g(x) = 1 \) for all \( x \) then \( f \in \text{UB} \) but \( g \notin \text{UB} \).

Using the version of Rice’s Theorem that distinguishes based on exact I/O behavior, for any \( f, g \) for which \( f(x) = g(x) \) for all \( x \),

\[
f \in \text{UB} \iff \exists x \forall y > x, f(x) = 0 \text{ or } f(x) \uparrow
\]

\[
\iff \exists x \forall y > x, g(x) = 0 \text{ or } g(x) \uparrow
\]

\[
\iff g \in \text{UB}
\]

which shows that if two partial functions have the same I/O behavior, they must both be in UB or both be out of UB.

3. Show that given a Semi–Thue system, \( S \), you can produce a Post Normal System, \( N_S \), such that \( x \Rightarrow^* y \) iff \( x \Rightarrow^*_S y \). You must give the construction of \( N_S \) from \( S \) and a justification of why this meets the condition stated above.

Given a Semi–Thue system \( S = (\Sigma, R) \) where

\[
R = \{ \alpha_1 \rightarrow \beta_1, \alpha_2 \rightarrow \beta_2, \ldots, \alpha_k \rightarrow \beta_k \}
\]

with \( \alpha_i, \beta_i \in \Sigma^* \) for \( 1 \leq i \leq k \), we need to construct a Post Normal System \( N_S \) such that if \( x, y \in \Sigma^* \) then \( x \Rightarrow^*_S y \) iff \( x \Rightarrow^*_N y \).

Our construction will be \( N_S = (\Sigma_S, R_S) \) where

\[
\Sigma_S = \Sigma \cup \overline{\Sigma}
\]

and \( \overline{\Sigma} \) is everything in \( \Sigma \) but with a line over it. If \( w = a_1 a_2 \cdots a_k \in \Sigma^* \) then \( \overline{w} = \overline{a_1 a_2 \cdots a_k} \in \overline{\Sigma}^* \). Then let \( R_S \) be the union of the following normal rules:
1. \( \{ \alpha P \rightarrow P \beta : \alpha \rightarrow \beta \in R \} \)
2. \( \{ aP \rightarrow P \bar{a} : a \in \Sigma \} \)
3. \( \{ aP \rightarrow Pa : a \in \Sigma \} \).

We now need to show that this construction is correct. To show that \( x \Rightarrow^* y \) implies \( x \Rightarrow^* y \) we only need to show that \( x \Rightarrow y \) implies \( x \Rightarrow^* y \). If \( x \Rightarrow y \) then \( x = uav \) and \( y = u\beta v \) where \( \alpha \rightarrow \beta \in R \).

Here is why \( uav \Rightarrow^* u\beta v \):

\[
\begin{align*}
  & uav \Rightarrow_{N_S} \alpha v\bar{u} & \text{...}|u| \text{ applications of type 2. rules} \\
  & \alpha v\bar{u} \Rightarrow_{N_S} v\bar{u}\bar{\beta} & \text{...application of a rule of type 1.} \\
  & v\bar{u}\bar{\beta} \Rightarrow_{N_S} u\bar{\beta}v & \text{...}|v| \text{ applications of type 2. rules} \\
  & u\bar{\beta}v \Rightarrow_{N_S} u\beta v & \text{...} |u\beta v| \text{ applications of 3. rules}
\end{align*}
\]

Now we need that for \( x, y \in \Sigma^* \), \( x \Rightarrow^* y \) implies \( x \Rightarrow^* y \). We can’t use the same technique as for the other direction since any application of a rule in the post normal system brings us out of \( \Sigma^* \). Observe that we can move from any string of the form \( u\alpha v \) to a string \( wuv \) by some number of applications of rules of type 2. or 3. where \( u, v, w \in \Sigma^* \).

We will do induction on the number of times a rule of type 1. is applied in a derivation. We want to show that for \( x, y \in \Sigma^* \) if a derivation \( x \Rightarrow^* y \) uses \( n \) applications of rules of type 1. for any \( n \geq 0 \) then \( x \Rightarrow^* y \).

Base: \( n = 0 \) and \( x \Rightarrow^* y \) implies \( x = y \) so \( x \Rightarrow^* y \).

IH: If \( x, y \in \Sigma^* \) and \( x \Rightarrow^* y \) and uses \( n \geq 0 \) applications of rules of type 3. then \( x \Rightarrow^* y \).

IS: If \( x \Rightarrow^* y \) using \( n \geq 1 \) applications of rules of type 1. then let \( \gamma_i \) be the form after applying the \( i \)th rule of type 1, so

\[
x \Rightarrow_{N_S}^* \gamma_1 \Rightarrow_{N_S}^* \gamma_2 \Rightarrow_{N_S}^* \cdots \Rightarrow_{N_S}^* \gamma_{n-1} \Rightarrow_{N_S}^* \gamma_n \Rightarrow_{N_S}^* y.
\]

Now \( \gamma_{n-1} = uv\bar{\beta} \) which we can derive \( v\beta u \in \Sigma^* \) from by some applications of rules of type 2. or 3. So we have that \( x \Rightarrow_{N_S}^* \gamma_n \Rightarrow_{N_S}^* z \) where \( z = v\beta u \) using \( n - 1 \) type 1. rules. Apply the induction hypothesis to \( x \Rightarrow_{N_S}^* z \) so we have that \( x \Rightarrow_{N_S}^* z \). Let \( \gamma'_n \Rightarrow_{N_S} \gamma_n \) then \( z \Rightarrow_{N_S}^* \gamma'_n \) through some applications of type 2. and 3. rules. Now let \( \alpha \rightarrow \beta \in R \) be the rule applied from \( \gamma'_n \) to get \( \gamma_n \) which leads to \( y \). Then \( z \Rightarrow_{N_S} y \) by the production \( \alpha \rightarrow \beta \in R \) and thus \( x \Rightarrow_{N_S}^* z \Rightarrow_{N_S}^* y \).

So combining we have if \( x, y \in \Sigma^* \) then \( x \Rightarrow_{N_S}^* y \) iff \( y \Rightarrow_{N_S}^* y \).