COT 5310 Homework 6 Key
Fall 2007

1. Using reduction from the complement of the Halting Problem, show the undecidability of the problem to determine if an arbitrary partial recursive function, f, has a summation upper bound. This means that there is an M, such that the sum of all values in the range of f (repeats are added in and divergence just adds 0) is $\leq M$.

   The set HALT $= \{\langle f, x\rangle \ : \ f(x)\downarrow\}$, therefore CoHALT $= \{\langle f, x\rangle \ : \ f(x)\uparrow\}$. The set of partial functions f with a summation upper bound can be described as

   $$\text{UB} = \{f \ : \ \exists x \forall y \ [y > x \Rightarrow (f(y) = 0 \text{ or } f(y)\uparrow)]\}.$$

   or in other words, only finitely many of the outputs can be non-zero.

   To show that CoHALT $\leq_m$ UB we define $g(\langle f, x\rangle) = g_{f,x}$ as

   $$g_{f,x}(y) = \mu\ t\ [\text{STP}(f, x, t)] + 1.$$

   Then if $\langle f, x\rangle \in$ CoHALT, $g_{f,x}(y)\uparrow$ for all $y \in \mathbb{N}$ so $\text{dom}(g_{f,x}) = \varnothing \Rightarrow g_{f,x} \in$ UB.

   If $\langle f, x\rangle \notin$ CoHALT, $g_{f,x}(y) \geq 1$ for all $y \in \mathbb{N}$. This means that no summation upper bound exists and $g_{f,x} \notin$ UB.

2. Use one of the versions of Rice's Theorem to show the undecidability of the problem to determine if an arbitrary partial recursive function, f, has a summation upper bound. This means that there is an M, such that the sum of all values in the range of f (repeats are added in and divergence just adds 0) is $\leq M$.

   We can consider using Rice's Theorem because UB is a set of partial function indices.

   UB is non-trivial because if $f(x) = 0$ and $g(x) = 1$ for all x then $f \in$ UB but $g \notin$ UB.

   Using the version of Rice's Theorem that distinguishes based on exact I/O behavior, for any f, g for which $f(x) = g(x)$ for all x,

   $$\begin{aligned} f \in \text{UB} &\Leftrightarrow \exists x \forall y > x,\ f(x) = 0 \text{ or } f(x)\uparrow \\ &\Leftrightarrow \exists x \forall y > x,\ g(x) = 0 \text{ or } g(x)\uparrow \\ &\Leftrightarrow g \in \text{UB} \end{aligned}$$

   which shows that if two partial functions have the same I/O behavior, they must both be in UB or both be out of UB.

3. Show that given a Semi–Thue system, S, you can produce a Post Normal System, N_S, such that $x \underset{S}{\Rightarrow}^* y$ iff $x \underset{N_S}{\Rightarrow}^* y$. You must give the construction of N_S from S and a justification of why this meets the condition stated above.

   Given a Semi–Thue system $S = (\Sigma, R)$ where

   $$\begin{aligned} R = \{&\alpha_1 \to \beta_1, \\ &\alpha_2 \to \beta_2, \\ &\vdots \\ &\alpha_k \to \beta_k\} \end{aligned}$$

   with $\alpha_i, \beta_i \in \Sigma^*$ for $1 \leq i \leq k$, we need to construct a Post Normal System N_S such that if $x, y \in \Sigma^*$ then $x \underset{S}{\Rightarrow}^* y$ iff $x \underset{N_S}{\Rightarrow}^* y$.

   Our construction will be $N_S = (\Sigma_S, R_S)$ where

   $$\Sigma_S = \Sigma \cup \overline{\Sigma}$$

   and $\overline{\Sigma}$ is everything in Σ but with a line over it. If $w = a_1 a_2 \cdots a_k \in \Sigma^*$ then $\overline{w} = \bar{a_1}\bar{a_2}\cdots\bar{a_k} \in \overline{\Sigma}^*$. Then let R_S be the union of the following normal rules:

1. $\{\alpha P \to P\bar{\beta} \ : \ \alpha \to \beta \in R\}$
2. $\{aP \to P\bar{a} \ : \ a \in \Sigma\}$
3. $\{\bar{a}P \to Pa \ : \ a \in \Sigma\}$.

We now need to show that this construction is correct. To show that $x \underset{S}{\Rightarrow}^* y$ implies $x \underset{N_S}{\Rightarrow}^* y$ we only need to show that $x \underset{S}{\Rightarrow} y$ implies $x \underset{N_S}{\Rightarrow}^* y$. If $x \underset{S}{\Rightarrow} y$ then $x = u\alpha v$ and $y = u\beta v$ where $\alpha \to \beta \in R$. Here is why $u\alpha v \underset{N_S}{\Rightarrow}^* u\beta v$:

$$
\begin{aligned}
u\alpha v &\underset{N_S}{\Rightarrow} \alpha v\bar{u} && \ldots |u| \text{ applications of type 2. rules} \\
\alpha v\bar{u} &\underset{N_S}{\Rightarrow} v\bar{u}\bar{\beta} && \ldots \text{application of a rule of type 1.} \\
v\bar{u}\bar{\beta} &\underset{N_S}{\Rightarrow} \bar{u}\bar{\beta}\bar{v} && \ldots |v| \text{ applications of type 2. rules} \\
\bar{u}\bar{\beta}\bar{v} &\underset{N_S}{\Rightarrow} u\beta v. && \ldots |u\beta v| \text{ applications of 3. rules}
\end{aligned}
$$

Now we need that for $x, y \in \Sigma^*$, $x \underset{N_S}{\Rightarrow}^* y$ implies $x \underset{S}{\Rightarrow}^* y$. We can't use the same technique as for the other direction since any application of a rule in the post normal system brings us out of Σ^*. Observe that we can move from any string of the form $u\bar{v}w$ to a string wuv by some number of applications of rules of type 2. or 3. where $u, v, w \in \Sigma^*$.

We will do induction on the number of times a rule of type 1. is applied in a derivation. We want to show that for $x, y \in \Sigma^*$ if a derivation $x \underset{N_S}{\Rightarrow}^* y$ uses n applications of rules of type 1. for any $n \geq 0$ then $x \underset{S}{\Rightarrow}^* y$.

Base: $n = 0$ and $x \underset{N_S}{\Rightarrow}^* y$ implies $x = y$ so $x \underset{S}{\Rightarrow}^* x$.

IH: If $x, y \in \Sigma^*$ and $x \underset{N_S}{\Rightarrow}^* y$ and uses $n \geq 0$ applications of rules of type 3. then $x \underset{S}{\Rightarrow}^* y$.

IS: If $x \underset{N_S}{\Rightarrow}^* y$ using $n \geq 1$ applications of rules of type 1. then let γ_i be the form after applying the ith rule of type 1, so

$$x \underset{N_S}{\Rightarrow}^* \gamma_1 \underset{N_S}{\Rightarrow}^* \gamma_2 \underset{N_S}{\Rightarrow}^* \cdots \underset{N_S}{\Rightarrow}^* \gamma_{n-1} \underset{N_S}{\Rightarrow}^* \gamma_n \underset{N_S}{\Rightarrow}^* y.$$

Now $\gamma_{n-1} = u\bar{v}\bar{\beta}$ which we can derive $v\beta u \in \Sigma^*$ from by a some applications of rules of type 2. or 3. So we have that $x \underset{N_S}{\Rightarrow}^* \gamma_{n-1} \underset{N_S}{\Rightarrow}^* z$ where $z = v\beta u$ using $n-1$ type 1. rules. Apply the induction hypothesis to $x \underset{N_S}{\Rightarrow}^* z$ so we have that $x \underset{S}{\Rightarrow}^* z$. Let $\gamma_n' \underset{N_S}{\Rightarrow} \gamma_n$ then $z \underset{N_S}{\Rightarrow}^* \gamma_n'$ through some applications of type 2. and 3. rules. Now let $\alpha \to \beta \in R$ be the rule applied from γ_n' to get γ_n which leads to y. Then $z \underset{S}{\Rightarrow} y$ by the production $\alpha \to \beta \in R$ and thus $x \underset{S}{\Rightarrow}^* z \underset{S}{\Rightarrow} y$ implies $x \underset{S}{\Rightarrow}^* y$.

So combining we have if $x, y \in \Sigma^*$ then $x \underset{S}{\Rightarrow}^* y$ iff $y \underset{N_S}{\Rightarrow}^* y$.