COT 5310 Homework 5 Key
Fall 2007

1. Let INF $= \{f : \mathrm{dom}(f)$ is infinite$\}$ and NE $= \{f :$ there is a y such that $f(y)$ converges$\}$. Show that NE $\leq_m$ INF. Present the mapping and then explain why it works as desired. To do this define a total recursive function g such that index f is in NE iff $g(f)$ is in INF. Be sure to address both cases (f in and f not in).

   We need a mapping g such that $f \in \mathrm{NE} \Leftrightarrow g(f) \in \mathrm{INF}$. So we don't write $g(f)(x)$ we'll use the notation $g_f = g(f)$. Define g_f to be

   $$g_f(\langle x, t\rangle) = \mu\ z\ [\mathrm{STP}(x, f, t)].$$

   So $g_f(\langle x, t\rangle)$ returns 0 if STP(x, f, t) otherwise it diverges.

   If $f \in$ NE there exists a y such that $f(y)$ converges, meaning there exists a $\langle y, t\rangle$ such that STP(y, f, t) is true. Then since STP$(y, f, t) \Rightarrow$ STP$(y, f, t + k)$ for all $k \geq 0$, $g_f \in$ INF.

   If $f \notin$ NE then there does not exist a y such that $f(y)$ converges, meaning that $\mathrm{dom}(f) = \varnothing$. This means that for all $\langle y, t\rangle$, STP(y, f, t) is false. Since the domain of g_f in this case has size 0, $g_f \notin$ INF.

   An alternative and equally valid definition for g_f (with a different proof) is

   $$g_f(x) = \mu\ \langle y, t\rangle\ [\mathrm{STP}(y, f, t)].$$

2. Is INF $\leq_m$ NE? If you say yes, show it. If you say no, give a convincing argument that INF is more complex than NE.

   One convincing argument that INF is more complex than NE is to look at their quantified definitions in terms of STP. For NE this is

   $$\mathrm{NE} = \{f\ :\ \exists\langle y, t\rangle\ \ \mathrm{STP}(y, f, t)\}\,.$$

   For the case of INF we need that for all x there exists a $y > x$ such that $f(y)$ converges.

   $$\mathrm{INF} = \{f\ :\ \forall x\ \exists\langle y, t\rangle\ \ y > x \text{ and } \mathrm{STP}(y, f, t)\}$$

   So from this we'd suspect that NE is recursively enumerable non–recursive and that INF is not recursively enumerable. However, there could conceivably be a better quantified expression for INF that doesn't need the $\forall$. But, because we thought long and hard, there probably isn't a better expression.

   For a proof, we could show that TOTAL $\leq_m$ INF by

   $$g_f(x) = \mu\ (y < x)[0 \cdot (\mu\ t\ \mathrm{STP}(f, y, t))]$$

   so that $g_f(x)$ converges (and returns x) only if $f(y)$ converges for all $y < x$. Then we need to verify that $f \in \mathrm{TOTAL} \Leftrightarrow g_f \in \mathrm{INF}$. It can also be shown that TOTAL $=_m$ INF by showing that INF $\leq_m$ TOTAL. But to avoid monopolizing all of the fun, this and the verification of TOTAL $\leq_m$ INF are left for the reader.

3. What if anything does Rice's Theorem have to say about the following? In each case explain by either showing that all of Rice's conditions are met or convincingly that at least one is not met.

   a.) RANGE $= \{f : $ there is a g such that $\mathrm{range}(g) = \mathrm{dom}(f)\}$
   b.) PRIMITIVE $= \{f : f$'s description uses no unbounded μ operations$\}$
   c.) FINITE $= \{f : \mathrm{dom}(f)$ is finite$\}$

It is clear that each of these is a question about sets of function indices.

a.) RANGE $= \{f \ : \ \text{there is a } g \text{ such that range}(g) = \text{dom}(f)\}$
This is a trivial property because for any function f, $g_f(x) = f(x) - f(x) + x$ has the property that $\text{range}(g_f) = \text{dom}(f)$.

b.) PRIMITIVE $= \{f \ : \ f\text{'s description uses no unbounded } \mu \text{ operations}\}$
This is not an I/O property because if we let $f(x) = 0$ and $g(x) = \mu\, z[1]$ then $f(x) = g(x)$ for all $x \in \mathbb{N}$ but $f \in$ PRIMITIVE while $g \notin$ PRIMITIVE.

c.) FINITE $= \{f \ : \ \text{dom}(f) \text{ is finite}\}$
The two functions $f(x) = 0$ and $g(x) = \mu\, z[0]$ demonstrate non–triviality because $\text{dom}(f) = \mathbb{N} \Rightarrow f \notin$ FINITE while $\text{dom}(g) = \varnothing \Rightarrow g \in$ FINITE.
This is not an I/O property (using the version which needs that $\text{dom}(f) = \text{dom}(g)$) because given two f, g such that $\text{dom}(f) = \text{dom}(g)$, $f \in \text{FINITE} \Leftrightarrow |\text{dom}(f)| \in \mathbb{N} \Leftrightarrow |\text{dom}(g)| \in N \Leftrightarrow g \in$ FINITE. Therefore Rice's Theorem applies and FINITE is undecidable.