1. Let \( \operatorname{INF} = \{ f : \text{dom}(f) \text{ is infinite} \} \) and \( \operatorname{NE} = \{ f : \text{there is a } y \text{ such that } f(y) \text{ converges} \} \). Show that \( \operatorname{NE} \leq_m \operatorname{INF} \). Present the mapping and then explain why it works as desired. To do this define a total recursive function \( g \) such that index \( f \) is in \( \operatorname{NE} \) iff \( g(f) \) is in \( \operatorname{INF} \). Be sure to address both cases (\( f \) in and \( f \) not in).

We need a mapping \( g \) such that \( f \in \operatorname{NE} \iff g(f) \in \operatorname{INF} \). So we don’t write \( g(f)(x) \) we'll use the notation \( g_f = g(f) \). Define \( g_f \) to be

\[
 g_f(\langle x, t \rangle) = \mu z [\operatorname{STP}(x, f, t)] .
\]

So \( g_f(\langle x, t \rangle) \) returns 0 if \( \operatorname{STP}(x, f, t) \) otherwise it diverges.

If \( f \in \operatorname{NE} \) there exists a \( y \) such that \( f(y) \) converges, meaning there exists a \( \langle y, t \rangle \) such that \( \operatorname{STP}(y, f, t) \) is true. Then since \( \operatorname{STP}(y, f, t) \Rightarrow \operatorname{STP}(y, f, t + k) \) for all \( k \geq 0 \), \( g_f \in \operatorname{INF} \).

If \( f \notin \operatorname{NE} \) then there does not exist a \( y \) such that \( f(y) \) converges, meaning that \( \text{dom}(f) = \emptyset \). This means that for all \( \langle y, t \rangle \), \( \operatorname{STP}(y, f, t) \) is false. Since the domain of \( g_f \) in this case has size 0, \( g_f \notin \operatorname{INF} \).

An alternative and equally valid definition for \( g_f \) (with a different proof) is

\[
 g_f(x) = \mu \langle y, t \rangle [\operatorname{STP}(y, f, t)] .
\]

2. Is \( \operatorname{INF} \leq_m \operatorname{NE} \)? If you say yes, show it. If you say no, give a convincing argument that \( \operatorname{INF} \) is more complex than \( \operatorname{NE} \).

One convincing argument that \( \operatorname{INF} \) is more complex than \( \operatorname{NE} \) is to look at their quantified definitions in terms of \( \operatorname{STP} \). For \( \operatorname{NE} \) this is

\[
 \operatorname{NE} = \{ f : \exists \langle y, t \rangle \ \operatorname{STP}(y, f, t) \} .
\]

For the case of \( \operatorname{INF} \) we need that for all \( x \) there exists a \( y > x \) such that \( f(y) \) converges.

\[
 \operatorname{INF} = \{ f : \forall x \exists \langle y, t \rangle \ y > x \text{ and } \operatorname{STP}(y, f, t) \}
\]

So from this we’d suspect that \( \operatorname{NE} \) is recursively enumerable non–recursive and that \( \operatorname{INF} \) is not recursively enumerable. However, there could conceivably be a better quantified expression for \( \operatorname{INF} \) that doesn’t need the \( \forall \). But, because we thought long and hard, there probably isn’t a better expression.

For a proof, we could show that \( \operatorname{TOTAL} \leq_m \operatorname{INF} \) by

\[
 g_f(x) = \mu (y < x)[0 : (\mu t \operatorname{STP}(f, y, t))]
\]

so that \( g_f(x) \) converges (and returns \( x \)) only if \( f(y) \) converges for all \( y < x \). Then we need to verify that \( f \in \operatorname{TOTAL} \iff g_f \in \operatorname{INF} \). It can also be shown that \( \operatorname{TOTAL} =_m \operatorname{INF} \) by showing that \( \operatorname{INF} \leq_m \operatorname{TOTAL} \). But to avoid monopolizing all of the fun, this and the verification of \( \operatorname{TOTAL} \leq_m \operatorname{INF} \) are left for the reader.

3. What if anything does Rice’s Theorem have to say about the following? In each case explain by either showing that all of Rice’s conditions are met or convincingly that at least one is not met.

a.) \( \operatorname{RANGE} = \{ f : \text{there is a } g \text{ such that range}(g) = \text{dom}(f) \} \)
b.) \( \operatorname{PRIMITIVE} = \{ f : f \text{'s description uses no unbounded } \mu \text{ operations} \} \)
c.) \( \operatorname{FINITE} = \{ f : \text{dom}(f) \text{ is finite} \} \)
It is clear that each of these is a question about sets of function indices.

a.) \( \text{RANGE} = \{ f : \text{there is a } g \text{ such that } \text{range}(g) = \text{dom}(f) \} \)

This is a trivial property because for any function \( f \), \( g_f(x) = f(x) - f(x) + x \) has the property that \( \text{range}(g_f) = \text{dom}(f) \).

b.) \( \text{PRIMITIVE} = \{ f : f\text{'s description uses no unbounded } \mu \text{ operations} \} \)

This is not an I/O property because if we let \( f(x) = 0 \) and \( g(x) = \mu z[1] \) then \( f(x) = g(x) \) for all \( x \in \mathbb{N} \) but \( f \in \text{PRIMITIVE} \) while \( g \notin \text{PRIMITIVE} \).

c.) \( \text{FINITE} = \{ f : \text{dom}(f) \text{ is finite} \} \)

The two functions \( f(x) = 0 \) and \( g(x) = \mu z[0] \) demonstrate non-triviality because \( \text{dom}(f) = \mathbb{N} \Rightarrow f \notin \text{FINITE} \) while \( \text{dom}(g) = \emptyset \Rightarrow g \in \text{FINITE} \).

This is not an I/O property (using the version which needs that \( \text{dom}(f) = \text{dom}(g) \)) because given two \( f, g \) such that \( \text{dom}(f) = \text{dom}(g) \), \( f \in \text{FINITE} \Leftrightarrow |\text{dom}(f)| \in \mathbb{N} \Leftrightarrow |\text{dom}(g)| \in \mathbb{N} \Leftrightarrow g \in \text{FINITE} \). Therefore Rice’s Theorem applies and \( \text{FINITE} \) is undecidable.