COT 5310 Homework 3 Key

3. Show that prfs are closed under mutual recursion. That is, assuming F_1, F_2 and G_1 and G_2 are pr, show that H_1 and H_2 are, where

   $H_1(0, x) = F_1(x)$
   $H_1(y + 1, x) = G_1(y, x, H_2(y, x))$

   $H_2(0, x) = F_2(x)$
   $H_2(y + 1, x) = G_2(y, x, H_1(y, x))$.

   Define H iteratively as

   $H(0, x) = \langle F_1(x), F_2(x)\rangle$
   $H(y + 1, x) = \langle G_1(y, x, \langle H(y, x)\rangle_2), G_2(y, x, \langle H(y, x)\rangle_1)\rangle$.

   H is primitive recursive by construction since pairing, G_1, G_2, F_1, and F_2 are all primitive recursive. The claim we need to prove (via induction) is that $H(y, x) = \langle H_1(y, x), H_2(y, x)\rangle$ for all $y \geq 0$.

   Base : $H(0, x) = \langle F_1(x), F_2(x)\rangle = \langle H_1(0, x), H_2(0, x)\rangle$.

   Hypothesis : $H(y, x) = \langle H_1(y, x), H_2(y, x)\rangle$ for all $y \geq 0$.

   Step : Let $y \geq 0$. Then

   $$H(y + 1, x) = \langle G_1(y, x, \langle H(y, x)\rangle_2), G_2(y, x, \langle H(y, x)\rangle_1)\rangle \quad \ldots\text{by definition}$$
   $$H(y + 1, x) = \langle G_1(y, x, H_2(y, x)), G_2(y, x, H_1(y, x))\rangle \quad \ldots\text{by hypothesis}$$
   $$H(y + 1, x) = \langle H_1(y + 1, x), H_2(y + 1, x)\rangle \quad \ldots\text{definition of } H_1 \text{ and } H_2.$$

   Now, $H_1(y, x) = \langle H(y, x)\rangle_1$ so H_1 is primitive recursive, and $H_2(y, x) = \langle H(y, x)\rangle_2$ so H_2 is primitive recursive.