COT 5310 Homework 2 Key

(a) Present a Register Machine that computes FIB. Assume `R1` $= x$; at termination set `R2` $= 1$ if x is a member of the Fibonacci sequence and 0 if not.

1.	$\texttt{INC}_4\texttt{[2]}$	// set up `R3` $= 0 = l$ and `R4` $= 1 = h$ as the first two Fibonacci numbers
2.	$\texttt{DEC}_1\texttt{[3, 14]}$	// move input `R1` and `R3` $(= l)$ to `R5`, finish if can't decrement
3.	$\texttt{DEC}_3\texttt{[4, 5]}$	
4.	$\texttt{INC}_5\texttt{[2]}$	
5.	$\texttt{INC}_1\texttt{[6]}$	// `R1` is now $x - l - 1$, make it equal $x - l$, this implies `R3`=0 now
6.	$\texttt{DEC}_4\texttt{[7, 9]}$	// move `R4`$(= h)$ into `R3` and copy to `R6`
7.	$\texttt{INC}_3\texttt{[8]}$	
8.	$\texttt{INC}_6\texttt{[6]}$	
9.	$\texttt{DEC}_5\texttt{[10, 12]}$	// now `R5` $= l$, move to `R4` and `R1`, thus restoring `R1` to x
10.	$\texttt{INC}_1\texttt{[11]}$	
11.	$\texttt{INC}_4\texttt{[9]}$	
12.	$\texttt{DEC}_6\texttt{[13, 2]}$	// move copy of h in `R6` to `R4` which contains h right now
13.	$\texttt{INC}_4\texttt{[12]}$	
14.	$\texttt{DEC}_3\texttt{[16, 15]}$	// if `R3` $= 0$ then $x \in$ FIB
15.	$\texttt{INC}_2\texttt{[16]}$	
16.	$\texttt{DEC}_3\texttt{[16, 17]}$	// begin cleanup to make things look pretty
17.	$\texttt{DEC}_4\texttt{[17, 18]}$	
18.	$\texttt{DEC}_5\texttt{[18, 19]}$	

(b) Present a Factor Replacement System that computes FIB. Assume starting number is $3^x \cdot 5$; at termination, result is $2 = 2^1$ if x is a member of the Fibonacci sequence; $1 = 2^0$ otherwise. Actually, it can be done without the 5, but that may make it easier.

The following FRS computes FIB. It uses the primes 2, 3, 5, 7, 11, 13, and 17. The first prime 2 stores the result, 3 stores the input, 5 represents the state, 7 is used to hold the current low Fibonacci number called l, 11 is used to hold the current high Fibonacci number called h, finally 13 and 17 are temporary variables. State 5^2 compares the input to the low Fibonacci number and 5^3 computes the next in the series. The list of values to the right of a rule represent the input before that rule is applied.

$$5^4 \cdot 7x \to 5^4 x$$
$$5^4 \cdot 11x \to 5^4 x$$
$$5^4 \cdot 13x \to 5^4 x$$
$$5^4 x \to x$$

cleanup

Rule	**2**	**3**	**5**	**7**	**11**	**13**	**17**
$5^3 \cdot 11 \cdot 13x \to 5^3 \cdot 7 \cdot 13 \cdot 17x$	0	$x-l$	3	0	h	l	0
$5^3 \cdot 13x \to 3 \cdot 5^3 \cdot 17x$	0	$x-l$	3	h	0	l	h
$5^3 \cdot 17x \to 5^3 \cdot 11x$	0	x	3	h	0	0	$h+l$
$5^3 x \to 5^2 x$	0	x	3	h	$h+l$	0	0

Rule	**2**	**3**	**5**	**7**	**11**	**13**	**17**
$3 \cdot 5^2 \cdot 7x \to 5^2 \cdot 13x$	0	x	2	l	h	0	0
$3 \cdot 5^2 x \to 3 \cdot 5^3 x$	0	$x-l$	2	0	h	l	0
$5^2 \cdot 7x \to 5^4 x$	0	0	2	$l-x$	h	x	0
$5^2 x \to 2 \cdot 5^4 x$	0	0	2	0	h	l	0

$$3 \cdot 5x \to 3 \cdot 5^2 \cdot 7 \cdot 11x$$
$$5x \to 2x$$

initialize and check for 0

(c) Prove that non-deterministic FRS's are no more powerful than non-deterministic VAS. This means you need only show that any non-deterministic FRS can be simulated by a non-deterministic VAS. Note: To do this most effectively, you need to first develop the notion of an instantaneous description (ID) of an FRS (that's a point in 1-space) and of a VAS (that's a point in n-space). You then need a mapping from an FRS ID to a corresponding VAS ID, and this mapping needs to be some function (many-one into), f. Next, there must be a mapping from the rules of the FRS to create those of the VAS, such that a single step of the FRS from x to y is mimicked by some finite number of of steps of the VAS from $f(x)$ to $f(y)$, where $f(y)$ is the first ID derived from the $f(x)$ that is a mapping from some ID of the VAS.

An instantaneous description of an FRS is a positive integer $x \in \mathbb{N}^+$. An instantaneous description of a VAS of n dimensions is a vector $v \in \mathbb{N}^n$. We will give the mapping and then argue its correctness.

Any non-deterministic FRS R can be represented as a set of 2-tuples over $\mathbb{N}^+$ where (a, b) corresponds to the rule $ax \to bx$. Let m be the index of the largest prime used in any side of any rule, and $k = |R|$. Then define the mapping $f\colon \mathbb{N}^+ \to \mathbb{N}^{m+k+1}$ by

$$f(x) = f(p_1^{x_1} \cdot p_2^{x_2} \cdots p_m^{x_m}) = \langle\, \underbrace{x_1, x_2, \ldots, x_m}_{m}, \underbrace{0, \ldots, 0}_{k}, 1 \,\rangle\,.$$

The first m dimensions contain the exponents for the prime decomposition of x, the next k dimensions will be used to specify which rule is being simulated, and the last dimension specifies which side of the rule to simulate (left if equal to 1 or right if equal to 0).

The vectors of the non-deterministic VAS will each have dimension $n = m + k + 1$. For the rule $a \to b$ with index r where $0 \le r \le k - 1$ add the following two vectors to the VAS:

$$\begin{array}{rcccccccccl} v_a = \langle & -a_1, & -a_2, & \ldots, & -a_m, & 0, & \ldots, & 1, & \ldots, & 0, & -1 \;\rangle \\ v_b = \langle & b_1, & b_2, & \ldots, & -a_m, & 0, & \ldots, & \overbrace{-1,}^{} & \ldots, & 0, & 1 \;\rangle \\ & & & & & & & m + r\,. & & & \end{array}$$

The -1 in the last position ensures that some b-vector must be chosen after using v_a, and the fact that only v_b has -1 at position $m + r$ ensures that it's the only b-vector that can be used.

To show the simulation is correct, it suffices to show that given two FRS ID's $x, y \in \mathbb{N}^+$ then $x \Rightarrow^* y$ if and only if $f(x) \Rightarrow^* f(y)$. The $x \Rightarrow^* y$ part is obtained by observing that $x \Rightarrow y$ implies $f(x) \Rightarrow^2 f(y)$, meaning that any application of an FRS rule can be simulated in two steps by the VAS. On the other hand, given an FRS ID x, $f(x) \Rightarrow^2 f(y)$ implies that $x \Rightarrow y$ because any two steps in the VAS because first some v_a must be chosen where a matches x, and as noted before this must be followed by its unique v_b. Finally, if $f(x) \Rightarrow z$ in a single step, then $f^{-1}(z)$ is empty and therefore cannot be an FRS ID because its last position is 0. Therefore $f(x) \Rightarrow^* f(y)$ implies $x \Rightarrow y$, and the simulation is correct.