4. Construct the natural cubic spline for the following data:

a. \( f(0) = 1, f(0.5) = 2.71828 \)

Since we are constructing a natural cubic spline on a single interval, we are dealing with a degenerate case: The linear interpolation between these two points will satisfy the necessary conditions. The linear interpolation is

\[
S(x) = 1 + 3.43656x
\]

which is thus our answer.

b. \( f(-0.25) = 1.33203, f(0.25) = 0.800781 \)

Again, our answer will be the linear interpolation between the two points. This linear interpolation is

\[
1.33203 - 1.062498(x + 0.25)
\]

c. \( f(0.1) = -0.29004996, f(0.2) = -0.56079734, f(0.3) = -0.81401972 \)

We are now constructing a cubic spline on two intervals. On the interval \([0.1, 0.2]\) we have the cubic

\[
S_0(x) = a_0 + b_0(x - 0.1) + c_0(x - 0.1)^2 + d_0(x - 0.1)^3
\]

and on the interval \([0.2, 0.3]\) we have the cubic

\[
S_1(x) = a_1 + b_1(x - 0.2) + c_1(x - 0.2)^2 + d_1(x - 0.2)^3
\]

We get the following four constraints from the condition that the splines must agree with the data at the nodes.

\[
a_0 = -0.29004996
\]

\[
a_1 = -0.56079734
\]

\[
a_0 + 0.1b_0 + 0.01c_0 + 0.001d_0 = -0.56079734
\]

\[
a_1 + 0.1b_1 + 0.01c_1 + 0.001d_1 = -0.81401972
\]

Two more constraints come from the conditions that \(S'_0(0.2) = S'_1(0.2)\) and \(S''_0(0.2) = S''_1(0.2)\). These are

\[
b_0 + 0.2c_0 + 0.03d_0 - b_1 = 0
\]

\[
2c_0 + 0.6d_0 - 2c_1 = 0
\]

The final two constraints come from the natural boundary conditions. They are

\[
c_0 = 0
\]

\[
2c_1 + 0.6d_1 = 0
\]

Solving this system of equations yields the spline

\[
S(x) = \begin{cases} 
-0.29004996 - 2.7512863(x - 0.1) + 4.38125(x - 0.1)^3 & : x \in [0.1, 0.2] \\
-0.56079734 - 2.6198488(x - 0.2) + 1.314375(x - 0.2)^2 - 4.38125(x - 0.2)^3 & : x \in [0.2, 0.3]
\end{cases}
\]
d. \( f(-1) = 0.86199480, f(-0.5) = 0.95802009, f(0) = 1.0986123, f(0.5) = 1.2943767 \)

We have \( n = 3, h_0 = h_1 = h_2 = 0.5, a_0 = 0.86199480, a_1 = 0.95802009, a_2 = 1.0986123, a_3 = 1.2943767 \). So the matrix \( A \) and the vectors \( \mathbf{b} \) and \( \mathbf{x} \) given in Theorem 3.1 have the forms

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0.5 & 2 & 0.5 & 0 \\
0 & 0.5 & 2 & 0.5 \\
0 & 0 & 0 & 0.5
\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}
0 \\
0.26740152 \\
0.33103314 \\
0
\end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3
\end{bmatrix}
\]

Solving the corresponding system of equations gives us

\[
c_0 = 0, \ c_1 = 0.09847639, \ c_2 = 0.14089747, \ c_3 = 0
\]

Solving for the remaining constants gives

\[
\begin{align*}
b_0 &= \frac{1}{h_0} (a_1 - a_0) - \frac{h_0}{3} (c_1 + 2c_0) \approx 0.1756378 \\
b_1 &= \frac{1}{h_1} (a_2 - a_1) - \frac{h_1}{3} (c_2 + 2c_1) \approx 0.2248760 \\
b_2 &= \frac{1}{h_2} (a_3 - a_2) - \frac{h_2}{3} (c_3 + 2c_2) \approx 0.3445630 \\
d_0 &= \frac{1}{3h_0} (c_1 - c_0) \approx 0.0656509 \\
d_1 &= \frac{1}{3h_1} (c_2 - c_1) \approx 0.0282807 \\
d_2 &= \frac{1}{3h_2} (c_3 - c_2) \approx -0.0939316
\end{align*}
\]

Yielding the natural cubic spline

\[
S(x) = \begin{cases}
0.86199480 + 0.1756378 (x + 1) + 0.0656509 (x + 1)^2 & : x \in [-1, -0.5] \\
0.95802009 + 0.2248760 (x + 0.5) + 0.0282807 (x + 0.5)^2 & : x \in [-0.5, 0] \\
1.0986123 + 0.3445630 (x + 0.5)^2 - 0.0939316 x^3 & : x \in [0, 0.5]
\end{cases}
\]

16. Construct a natural cubic spline to approximate \( f(x) = e^{-x} \) by using the values given by \( f(x) \) at \( x = 0, 0.25, 0.5, 0.75, \) and 1. Integrate the spline over \([0, 1]\) and compare the result to the real integral. Use the derivatives of the spline to approximate \( f'(0.5) \) and \( f''(0.5) \). Compare the approximations to the actual values.

We have \( n = 4, h_0 = h_1 = h_2 = h_3 = 0.25, a_0 = e^0 = 1, a_1 = e^{-0.25} \approx 0.778801, a_2 = e^{-0.5} \approx 0.606531, a_3 = e^{0.75} \approx 0.472367, a_4 = e^{-1} \approx 0.367880 \). So the matrix \( A \) and the vectors \( \mathbf{b} \) and \( \mathbf{x} \) given in Theorem 3.1 have the forms

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0.25 & 1 & 0.25 & 0 & 0 \\
0 & 0.25 & 1 & 0.25 & 0 \\
0 & 0 & 0.25 & 1 & 0.25 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad \mathbf{b} \approx \begin{bmatrix}
0 \\
0.587149 \\
1.457272 \\
0.356124 \\
0
\end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix}
\]

Solving the corresponding system of equations gives us

\[
c_0 = 0, \ c_1 = 0.523877, \ c_2 = 0.253090, \ c_3 = 0.292851, \ c_4 = 0
\]
Solving for the remaining constants gives

\[ b_0 = \frac{1}{h_0} (a_1 - a_0) - \frac{h_0}{3} (c_1 + 2c_0) \approx -0.928453 \]

\[ b_1 = \frac{1}{h_1} (a_2 - a_1) - \frac{h_1}{3} (c_2 + 2c_1) \approx -0.797484 \]

\[ b_2 = \frac{1}{h_2} (a_3 - a_2) - \frac{h_2}{3} (c_3 + 2c_2) \approx -0.603242 \]

\[ b_3 = \frac{1}{h_3} (a_4 - a_3) - \frac{h_3}{3} (c_4 + 2c_3) \approx -0.466757 \]

\[ d_0 = \frac{1}{3h_0} (c_1 - c_0) \approx 0.698502 \]

\[ d_1 = \frac{1}{3h_1} (c_2 - c_1) \approx -0.361048 \]

\[ d_2 = \frac{1}{3h_2} (c_3 - c_2) \approx 0.053015 \]

\[ d_3 = \frac{1}{3h_3} (c_4 - c_3) \approx -0.390468 \]

Yielding the natural cubic spline

\[ S(x) = \begin{cases} 
1 - 0.928453x + 0.698502x^3 & : x \in [0, 0.25] \\
0.778801 - 0.797484 (x - 0.25) + 0.523877 (x - 0.25)^2 - 0.361048 (x - 0.25)^3 & : x \in [0.25, 0.5] \\
0.606531 - 0.603242 (x - 0.5) + 0.253090 (x - 0.5)^2 + 0.053015 (x - 0.5)^3 & : x \in [0.5, 0.75] \\
0.472367 - 0.466757 (x - 0.75) + 0.292851 (x - 0.75)^2 - 0.090262 (x - 0.75)^3 & : x \in [0.75, 1] 
\end{cases} \]

Integrating each of the cubic polynomials that make up the spline yields

\[ t(x) = \begin{cases} 
x - 0.464227x^2 + 0.174626x^4 & : x \in [0, 0.25] \\
0.778801x - 0.397424 (x + 0.25)^2 + 0.174626 (x - 0.25)^3 - 0.590262 (x - 0.25)^4 & : x \in [0.25, 0.5] \\
0.606531x - 0.301621 (x - 0.5)^2 + 0.084363 (x - 0.5)^3 + 0.013254 (x - 0.5)^4 & : x \in [0.5, 0.75] \\
0.472367x - 0.233379 (x - 0.75)^2 + 0.097617 (x - 0.75)^3 - 0.097617 (x - 0.75)^4 & : x \in [0.75, 1] 
\end{cases} \]

Computing the definite integral over each of our subintervals and adding them all up yields

\[ \int_0^1 e^{-x} \approx 0.632624 \]

The actual value of this integral is

\[ \int_0^1 e^{-x} = 1 - \frac{1}{e} \approx 0.632121 \]

So we see that the approximation is good to three decimal places.

Since the first and second derivatives of the components of the spline agree on the nodes. We can approximate the first and second derivatives of \( e^{-x} \) by computing the first and second derivatives of the cubic on \([0.25, 0.5]\) at 0.5.

The first derivative of the cubic on \([0.25, 0.5]\) is

\[-0.797384 + 1.047754 (x - 0.25) - 1.083144 (x - 0.25)^2 \]

Evaluating this polynomial at 0.5 gives the approximation

\[ \left. \frac{d}{dx} e^{-x} \right|_{x=0.5} \approx -0.603142 \]
The actual derivative at this point is
\[
\frac{d}{dx} e^{-x} \bigg|_{x=0.5} = -e^{-x} \bigg|_{x=0.5} \approx -0.606531
\]

The spline gives a good approximation for the first derivative. The second derivative of the cubic on \([0.25, 0.5]\) is
\[
1.047754 - 2.166288 (x - 0.25)
\]
Evaluating at 0.5 yields the approximation
\[
\frac{d^2}{dx^2} e^{-x} \bigg|_{x=0.5} \approx 0.506182
\]

The actual second derivative at 0.5 is
\[
\frac{d^2}{dx^2} e^{-x} \bigg|_{x=0.5} = e^{-x} \bigg|_{x=0.5} \approx 0.606531
\]

The cubic spline does not give a good approximation for the second derivative. This makes sense since the construction of the spline doesn’t take the derivatives of any order of the function we seek to approximate into consideration. In general, the spline would not give a good approximation for the first derivative either, but it does for certain well behaved functions.