Numerical Calculus (COT 4500) Final Exam Solutions 4/27/2012

1) a) $\min = \dfrac{252}{.0014} = \dfrac{2520000}{14} = \boxed{180{,}000}$ (2 pts)

$\max = \dfrac{252}{.0006} = \dfrac{2520000}{6} = \boxed{420{,}000}$ (3 pts)

b) $47.625 = 2^5 + 2^3 + 2^2 + 2^1 + 2^0 + 2^{-1} + 2^{-3}$
$= 2^5(1 + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5} + 2^{-6} + 2^{-8})$

Thus, $\underbrace{\boxed{s=0}}_{1\text{ pt}}$, $\underbrace{\boxed{c=1028}}_{1\text{ pt}}$, $\underbrace{\boxed{f = 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5} + 2^{-6} + 2^{-8}}}_{1\text{ pt}}$

The first 20 bits of the representation are:

$\underbrace{\boxed{0\ 10000000100\ 01111101}}_{2\text{ pts}}$, the rest are 0.

2) Bisection yields the following:

Iteration	x	f(x)
1	2.5	.375
2	2.25	-0.546875
3	2.375	-0.150390625
4	2.4375	0.095458984
5	2.40625	-0.031585693

$\boxed{\text{1 pt each value}}$

3) Newton's method yields the following (using $f'(x) = 3x^2 - 6x + 1$):

Iteration	x	f(x)
1	3	4
2	2.6	0.896
3	2.442253521	0.115517612
4	2.415010637	0.003190993
5	2.414214235	0.000002692

$\boxed{\text{1 pt each value}}$

4) $\tan(.25) = .26$, $\tan(.75) = .93$, $\tan(1.25) = 3.01$

$$L_0(x) = \frac{(x-.75)(x-1.25)}{(.25-.75)(.25-1.25)} = 2\left(x-\tfrac{3}{4}\right)\left(x-\tfrac{5}{4}\right) \quad 2\text{pts}$$

$$L_1(x) = \frac{(x-.25)(x-1.25)}{(.75-.25)(.75-1.25)} = -4\left(x-\tfrac{1}{4}\right)\left(x-\tfrac{5}{4}\right) \quad 2\text{pts}$$

$$L_2(x) = \frac{(x-.25)(x-.75)}{(1.25-.25)(1.25-.75)} = 2\left(x-\tfrac{1}{4}\right)\left(x-\tfrac{3}{4}\right) \quad 2\text{pts}$$

$$P(x) = (.26)2\left(x-\tfrac{3}{4}\right)\left(x-\tfrac{5}{4}\right)$$
$$-(.93)4\left(x-\tfrac{1}{4}\right)\left(x-\tfrac{5}{4}\right)$$
$$+(3.01)2\left(x-\tfrac{1}{4}\right)\left(x-\tfrac{3}{4}\right) \quad 2\text{pts}$$

$$= .26(2x^2-4x+\overset{1.875}{\cancel{7.8}}) - .93(4x^2-6x+1.25)$$
$$+3.01(2x^2-2x+.375)$$

$$= .52x^2 - 1.04x + .4875 - 3.72x^2 + 5.58x - 1.1625$$
$$+ 6.02x^2 - 6.02x + 1.12875$$

$$= \boxed{2.82x^2 - 1.48x + 0.45375} \quad 2\text{pts}$$

5) (a) Slope between $f(x_0-h), f(x) = \dfrac{f(x)-f(x_0-h)}{h}$ 1 pts

Slope between $f(x_0+h), f(x) = \dfrac{f(x_0+h)-f(x_0)}{h}$ 1 pts

$$\text{Average} = \frac{\frac{f(x_0)-f(x_0-h)}{h} + \frac{f(x_0+h)-f(x_0)}{h}}{2}$$ 1 pt

$$= \frac{f(x_0)-f(x_0-h)+f(x_0+h)-f(x_0)}{2h}$$

$$= \boxed{\frac{f(x_0+h)-f(x_0-h)}{2h}}$$ 1 pts

Slope between $f(x_0+h), f(x_0-h) = \dfrac{f(x_0+h)-f(x_0-h)}{(x_0+h)-(x_0-h)}$

1 pts

1 pt $= \boxed{\dfrac{f(x_0+h)-f(x_0-h)}{2h}}$

By inspection, these two expressions are equal, as desired.

(b) Consider $f(x) = 125x^3 - 25x^2$ evaluated at $x = .1$, using $h = .1$. 2 pts

$f'(x) = 375x^2 - 50x$, so
$f'(.1) = 3.75 - 5 = -1.25$ 1 pt

Using the Three-Point Midpoint Formula, we approximate $f'(.1)$ as follows:

$$f'(.1) \sim \frac{f(.2)-f(0)}{.2} = \frac{0-0}{.2} = \boxed{0}$$ 1 pt

The error is 100%!

6) $\int_{.5}^{1.25} \sin(x^2)dx = \int_{.5}^{.75} \sin(x^2)dx + \int_{.75}^{1} \sin(x^2)dx + \int_{1}^{1.25} \sin(x^2)dx$

Using Simpson's Rule we approximate these integrals to be:

6pts

2pts $\left[\frac{.25}{6}\left[\sin(.5^2)+4\sin(.625^2)+\sin(.75^2)+\right.\right.$

$\sin(.75^2)+4\sin(.875^2)+\sin(1^2)+$

$\left.\left.\sin(1^2)+4\sin(1.125^2)+\sin(1.25^2)\right]\right] \sim \boxed{0.504463}$ 2pts

7) $y\sin(2x) + 2y'\sin^2 x = 2\sin(x)\cos(2x)$

$\frac{2y'\sin^2 x}{2\sin^2 x} + \frac{y\cdot 2\sin x\cos x}{2\sin^2 x} = \frac{2\sin x\cos 2x}{2\sin^2 x}$

$y' + y\cdot\frac{\cos x}{\sin x} = \frac{\cos 2x}{\sin x}$] 4pts

$I = e^{\int \frac{\cos x}{\sin x}dx} = e^{\int \frac{du}{u}} \quad u=\sin x \quad = e^{\ln u} = e^{\ln \sin x}$

$= \boxed{\sin x}$ 2pts

$\int (\sin x)y' + y\cdot(\cos x) = \int \cos 2x$

$y\sin x = \frac{1}{2}\sin 2x + C$] 1pt

$y\sin x = \frac{1}{2}\cdot 2\sin x\cos x + C$

$y\sin x = \sin x\cdot\cos x + C$

$y = \cos x + C\csc x$] 1pt

Plugging in $x=\frac{\pi}{4}$, $5\sqrt{2} = \cos\left(\frac{\pi}{4}\right) + C\csc\left(\frac{\pi}{4}\right)$

$5\sqrt{2} = \frac{\sqrt{2}}{2} + \sqrt{2}C$

SOLN $\boxed{y = \cos x + \frac{9}{2}\csc x}$ 2pts $\frac{9\sqrt{2}}{2} = \sqrt{2}C \Rightarrow \boxed{C = \frac{9}{2}}$

8) Here is the result of running Euler's Method:

i	t_i	w_i
1	2.4	1.7
2	2.8	4.123881
3	3.2	8.201563
4	3.6	14.349148
5	4.0	$\boxed{22.973879}$

1 pt each Value

9)

$$(1)\ 2a+b+c+d=4$$
$$(2)\ a+2b+c+d=2$$
$$(3)\ a+b+2c+d=6$$
$$2a+2b+2c+4d=11 \Rightarrow (4)\ a+b+c+2d=\frac{11}{2}$$

Adding, we get $5a+5b+5c+5d=\frac{35}{2}$,
so $a+b+c+d=\frac{7}{2}$.

Subtract this equation from each of the four labeled equations to yield
$\boxed{a=\frac{1}{2},\ b=-\frac{3}{2},\ c=\frac{5}{2},\ d=2}$

Full Credit if correct
1 pt off for each error

10) $$\sum_{i=0}^{n-1} \omega^{ij}\omega^{ik} = \sum_{i=0}^{n-1} \omega^{ij+ik} = \sum_{i=0}^{n-1} \omega^{i(j+k)}$$ 3pts

Let $C = j+k$ and note that $\gcd(C,n) = 1$. This sum is:

$$\omega^0 + \omega^C + \omega^{2C} + \omega^{3C} + \dots + \omega^{(n-1)C}.$$

Remember that $\omega^a = \omega^b$ iff $a \equiv b \pmod{n}$.

3pts Now, we will show that each term above is unique. Assume to the contrary that $\omega^{xC} = \omega^{yC}$ for integers x, y with $x \neq y$ and $0 \leq x, y \leq n-1$. It follows that

$xC \equiv yC \pmod{n}$

$xC - yC \equiv 0 \pmod{n}$

$C(x-y) \equiv 0 \pmod{n}$

1pt Since $\gcd(n, C) = 1$, it follows that $n \mid (x-y)$. But this contradicts $x \neq y$ and $|x-y| \leq n-1$.

10 con't) It follows that all listed terms are unique and equal to terms of the form ω^i, for $0 \leq i \leq n-1$. This is because there are n terms, all of which are unequal and there are only n possible values for the exponents. Thus, it follows that:

3pts $$\sum_{i=0}^{n-1} \omega^{ij}\omega^{ik} = \sum_{i=0}^{n-1} \omega^{iC} = \sum_{i=0}^{n-1} \omega^{i} = \boxed{0}.$$