COT 4500, Numerical Calculus — Assigned: January 28, 2009
S. Lang, Spring 2009 — **Assignment #2** (30 pts.) — Due: Friday, February.6 in class

Note that Question 6 added on 1/30/2008.

Instruction: Write your answers clearly and show all relevant work including details. You may use a calculator for the work.

Exercise Set for Section 1-2 (pp. 26 – 29):

1. (4 pts.) Find the value for $p = \sqrt[5]{242}$ using 5-digit rounding arithmetic. Find the (largest) interval in which an approximation p^* to p lies if (a) the absolute error is within 10^{-5}; and (b) the relative error is within 10^{-5}. Use 5-digit rounding arithmetic for this question.
2. (3 pts.) Perform the following calculation (a) exactly; (b) using 3-digit chopping arithmetic; (c) using 3-digit rounding arithmetic: $\frac{2}{11} \cdot \frac{11}{17}$. For Parts (b) and (c), calculate each fraction separately then multiply the results, using chopping or rounding arithmetic, respectively, in each step.
3. (3 pts.) The value e^{-1} can be computed by the formula: $e^{-1} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - + \ldots = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!}$.
   (This can be verified by using the Taylor series for $f(x) = e^x$ about the point $x_0 = 0$ then substituting $x = -1$.) Compute the absolute and relative error if e^{-1} is approximated by $e^{-1} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}$ where all calculations are done using 5-digit chopping arithmetic.
4. (8 pts.) Let $f(x) = \frac{e^{-2x} - 1}{x}$ for $x \neq 0$; and $f(0) = -2$. (It can be proven that $f(x)$ is infinitely differentiable for all x.) Answer each of the following questions:
   (a) Find $\lim_{x \to 0} f(x)$. (Hint: Use L'Hopital's rule.)
   (b) Use 3-digit chopping arithmetic to calculate $f(0.05)$. That is, use a calculator to calculate $e^{-2(0.05)} - 1$ using 3-digit chopping, then divide the result by 0.05 and report the answer using 3-digit chopping.
   (c) Replace $f(x)$ by its third-degree Maclaurin polynomial and redo Part (b).
   (d) The exact value for $f(0.05)$ is -1.90325 using 6-digit chopping. Find the absolute errors for the values computed in Parts (b) and (c).
5. (6 pts.) Given the following 64-bit floating-point value in IEEE 754-1985 format:
   1 00000110011 010101 (followed by 46 zeros).
   (a) Find the decimal value (and show the steps of your work).
   (b) Conversely, convert the answer of Part (a) back to the 64-bit floating-point format (and show the steps of your work).
6. (6 pts.) Suppose $f \in C^4[a-h, a+h]$ where $h > 0$ is a small number. An approximation formula for $f'(a)$ is $f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$ when h is sufficiently small.
   (a) Use Taylor's Theorem to find the error term in the above approximation. **Hint**: Apply Taylor's Theorem to find the second-degree Taylor polynomial plus the error term, $P_2(x) + R_2(x)$ for $f(x+h)$, about the point $x_0 = a$. Similarly, find the second-degree Taylor polynomial plus the error term, $Q_2(x) + S_2(x)$ for $f(x-h)$, about the point $x_0 = a$. Calculate $\frac{f(a+h) - f(a-h)}{2h}$ then show that $f'(a) = \frac{f(a+h) - f(a-h)}{2h} - \frac{h^2}{6} f^{(3)}(\xi)$ where $a-h < \xi < a+h$. Note that the Intermediate Value Theorem of the text is useful in the last step of formula derivation.

(b) If $h = 10^{-5}$ and $f(x) = \sin x$, where $a = 1.0$. Find the exact value for $f'(a)$ and the absolute error using the approximate value given above assuming 6-digit chopping arithmetic in the calculations.

Note on Question 1(a) of Assignment #1:

Consider the function $f(x) = \dfrac{\sin x}{x+1} - 0.4$ defined on $x \in (-1, \infty)$. Find the value for $\lim_{x\to\infty} f(x)$.

Answer: Since $|\sin x| \le 1$ and $\dfrac{1}{x+1} \to 0$ as $x \to \infty$, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(\dfrac{\sin x}{x+1} - 0.4) = 0 - 0.4 = -0.4$.