Chapter 5, Section 5.1

**Definition 5.1** A function \( f(t, y) \) is said to satisfy a Lipschitz condition in the variable \( y \) on a set \( D \subset \mathbb{R}^2 \) if a constant \( L > 0 \) exists with

\[
|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,
\]

whenever \( (t, y_1), (t, y_2) \in D \). The constant \( L \) is called a **Lipschitz constant** for \( f \).

**Definition 5.2** A set \( D \subset \mathbb{R}^2 \) is said to be convex if whenever \( (t_1, y_1) \) and \( (t_2, y_2) \) belong to \( D \) and \( \lambda \) is in \([0, 1]\), the point \((\lambda t_1 + (1 - \lambda) t_2, \lambda y_1 + (1 - \lambda) y_2)\) also belongs to \( D \).

**Theorem 5.3** Suppose \( f(t, y) \) is defined on a convex set \( D \subset \mathbb{R}^2 \). If a constant \( L > 0 \) exists with

\[
\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L, \quad \text{for all } (t, y) \in D,
\]

then \( f \) satisfies a Lipschitz condition on \( D \) in the variable \( y \) with Lipschitz constant \( L \).

**Theorem 5.4** Suppose that \( D = \{(t, y) \mid a \leq t \leq b, -\infty < y < \infty\} \) and that \( f(t, y) \) is continuous on \( D \). If \( f \) satisfies a Lipschitz condition on \( D \) in the variable \( y \), then the initial-value problem

\[
y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,
\]

has a unique solution \( y(t) \) for \( a \leq t \leq b \).

**Definition 5.5** The initial-value problem

\[
\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,
\]

is said to be a **well-posed problem** if:

- A unique solution, \( y(t) \), to the problem exists, and
- There exist constants \( \epsilon_0 > 0 \) and \( k > 0 \) such that for any \( \epsilon \), with \( \epsilon_0 > \epsilon > 0 \), whenever \( \delta(t) \) is continuous with \( |\delta(t)| < \epsilon \) for all \( t \) in \([a, b]\), and when \( |\delta_0| < \epsilon \), the initial-value problem

\[
\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \leq t \leq b, \quad z(a) = \alpha + \delta_0,
\]

has a unique solution \( z(t) \) that satisfies

\[
|z(t) - y(t)| < k\epsilon \quad \text{for all } t \in [a, b].
\]

**Theorem 5.6** Suppose \( D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\} \). If \( f \) is continuous and satisfies a Lipschitz condition in the variable \( y \) on the set \( D \), then the initial-value problem

\[
\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha
\]

is well-posed.

**Section 5.2**

Euler's method constructs \( w_i \approx y(t_i) \), for each \( i = 1, 2, \ldots, N \), by deleting the remainder term. Thus, Euler's method is

\[
w_0 = \alpha,
\]

\[
w_{i+1} = w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1. \tag{5.8}
\]

Equation (5.8) is called the **difference equation** associated with Euler's method.
**Theorem 5.9** Suppose $f$ is continuous and satisfies a Lipschitz condition with constant $L$ on

$$D = \{(t, y) \mid a \leq t \leq b, -\infty < y < \infty\}$$

and that a constant $M$ exists with

$$|y''(t)| \leq M, \quad \text{for all } t \in [a, b].$$

Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

and $w_0, w_1, \ldots, w_N$ be the approximations generated by Euler's method for some positive integer $N$. Then, for each $i = 0, 1, 2, \ldots, N$,

$$|y(t_i) - w_i| \leq \frac{hM}{2L} \left[ e^{L(t_i-a)} - 1 \right]. \quad (5.10)$$

**Theorem 5.10** Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \quad (5.12)$$

and $u_0, u_1, \ldots, u_N$ be the approximations obtained using Eq. (5.11). If $|\delta_i| < \delta$ for each $i = 0, 1, \ldots, N$ and the hypotheses of Theorem 5.9 hold for Eq. (5.12), then

$$|y(t_i) - u_i| \leq \frac{1}{L} \left( \frac{hM}{2} + \frac{\delta}{h} \right) \left[ e^{L(t_i-a)} - 1 \right] + |\delta_0| e^{L(t_i-a)}, \quad (5.13)$$

for each $i = 0, 1, \ldots, N$. □

**Section 5.3**

**Definition 5.11** The difference method

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1,$$

has local truncation error

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \ldots, N - 1$.

**Taylor method of order $n$:**

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1. \quad (5.17)$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!}f^{(n-1)}(t_i, w_i).$$

Note that Euler's method is Taylor's method of order one.
Theorem 5.12 If Taylor’s method of order \( n \) is used to approximate the solution to
\[
y'(t) = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = \alpha,
\]
with step size \( h \) and if \( y \in C^{n+1}[a, b] \), then the local truncation error is \( O(h^n) \).

Section 5.4

Theorem 5.13 Suppose that \( f(t, y) \) and all its partial derivatives of order less than or equal to \( n + 1 \) are continuous on \( D = \{(t, y) \mid a \leq t \leq b, c \leq y \leq d\} \), and let \((t_0, y_0) \in D\). For every \((t, y) \in D\), there exists \( \xi \) between \( t \) and \( t_0 \) and \( \mu \) between \( y \) and \( y_0 \) with
\[
f(t, y) = P_n(t, y) + R_n(t, y),
\]

where
\[
P_n(t, y) = f(t_0, y_0) + \left[ (t - t_0) \frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(t_0, y_0) \right]
+ \left[ \frac{(t - t_0)^2}{2} \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0) \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) \right]
+ \left[ \frac{(y - y_0)^2}{2} \frac{\partial^2 f}{\partial y^2}(t_0, y_0) \right] + \cdots
+ \left[ \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (t - t_0)^{n-j} (y - y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(t_0, y_0) \right]
\]

and
\[
R_n(t, y) = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t - t_0)^{n+1-j} (y - y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(\xi, \mu).
\]

The function \( P_n(t, y) \) is called the \textbf{\( n \)th Taylor polynomial in two variables} for the function \( f \) about \((t_0, y_0)\), and \( R_n(t, y) \) is the remainder term associated with \( P_n(t, y) \).

The difference-equation method resulting from replacing \( T^{(2)}(t, y) \) in Taylor’s method of order two by \( f(t + (h/2), y + (h/2)f(t, y)) \) is a specific Runge–Kutta method known as the \textbf{Midpoint method}.

\textbf{Midpoint Method}

\[
w_0 = \alpha,
\]
\[
w_{i+1} = w_i + hf\left( t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i) \right), \quad \text{for each } i = 0, 1, \ldots, N - 1.
\]
Two other $O(h^2)$ methods of the form: $\alpha f(t, y) + \alpha r(t, y) + \delta_2 f(t, y) \quad \Rightarrow [5.2]$  

Modified Euler Method

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + \frac{h}{2} [f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))],$$
for each $i = 0, 1, 2, \ldots, N - 1.$

Heun’s Method

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + \frac{h}{4} \left[ f(t_i, w_i) + 3f \left( t_i + \frac{2}{3}h, w_i + \frac{2}{3}hf(t_i, w_i) \right) \right],$$
for each $i = 0, 1, 2, \ldots, N - 1.$

Runge–Kutta Order Four:

$$w_0 = \alpha,$$
$$k_1 = hf(t_i, w_i),$$
$$k_2 = hf \left( t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1 \right),$$
$$k_3 = hf \left( t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2 \right),$$
$$k_4 = hf(t_{i+1}, w_i + k_3),$$
$$w_{i+1} = w_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4),$$
for each $i = 0, 1, \ldots, N - 1.$ This method has local truncation error $O(h^4)$ provided the solution $y(t)$ has five continuous derivatives.

Chapter 6:

**Theorem 6.5** Let $A, B,$ and $C$ be $n \times m$ matrices and $\lambda$ and $\mu$ be real numbers. The following properties of addition and scalar multiplication hold:

(a) $A + B = B + A,$
(b) $(A + B) + C = A + (B + C),$
(c) $A + O = O + A = A,$
(d) $A + (-A) = -A + A = 0,$
(e) $\lambda(A + B) = \lambda A + \lambda B,$
(f) $(\lambda + \mu)A = \lambda A + \mu A,$
(g) $\lambda(\mu A) = (\lambda \mu)A,$
(h) $1A = A.$
Theorem 6.9 Let $A$ be an $n \times m$ matrix, $B$ be an $m \times k$ matrix, $C$ be a $k \times p$ matrix, $D$ be an $m \times k$ matrix, and $\lambda$ be a real number. The following properties hold:

(a) $A(BC) = (AB)C$; \hspace{1cm} (Associative law for multiplication)
(b) $A(B + D) = AB + AD$; \hspace{1cm} (Distributive law)
(c) $I_n B = B$ and $B I_k = B$; \hspace{1cm} (Identity Element)
(d) $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

Definition 6.10 An $n \times n$ matrix $A$ is nonsingular (or invertible) if an $n \times n$ matrix $A^{-1}$ exists with $AA^{-1} = A^{-1}A = I$. The matrix $A^{-1}$ is called the \textit{inverse} of $A$. A matrix without an inverse is called singular (or noninvertible).

The following properties regarding matrix inverses follow from Definition 6.10. The proofs of these results are considered in Exercise 5.

Theorem 6.11 For any nonsingular $n \times n$ matrix $A$:

(a) $A^{-1}$ is unique.
(b) $A^{-1}$ is nonsingular and $(A^{-1})^{-1} = A$.
(c) If $B$ is also a nonsingular $n \times n$ matrix, then $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 6.13 The following operations involving the transpose of a matrix hold whenever the operation is possible:

(a) $(A^t)^t = A$.
(b) $(A + B)^t = A^t + B^t$.
(c) $(AB)^t = B^t A^t$.
(d) If $A^{-1}$ exists, then $(A^{-1})^t = (A^t)^{-1}$.

Definition 6.14 (a) If $A = [a]$ is a $1 \times 1$ matrix, then $\text{det} A = a$.

(b) If $A$ is an $n \times n$ matrix, the \textit{minor} $M_{ij}$ is the determinant of the $(n - 1) \times (n - 1)$ submatrix of $A$ obtained by deleting the $i$th row and $j$th column of the matrix $A$.

(c) The \textit{cofactor} $A_{ij}$ associated with $M_{ij}$ is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.

(d) The \textit{determinant} of the $n \times n$ matrix $A$, when $n > 1$, is given either by

$$\text{det} A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } i = 1, 2, \ldots, n,$$

or by

$$\text{det} A = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } j = 1, 2, \ldots, n.$$
**Theorem 6.15** Suppose $A$ is an $n \times n$ matrix:

(a) If any row or column of $A$ has only zero entries, then $\det A = 0$.

(b) If $A$ has two rows or two columns the same, then $\det A = 0$.

(c) If $\tilde{A}$ is obtained from $A$ by the operation $(E_i) \leftrightarrow (E_j)$, with $i \neq j$, then $\det \tilde{A} = -\det A$.

(d) If $\tilde{A}$ is obtained from $A$ by the operation $(\lambda E_i) \rightarrow (E_i)$, then $\det \tilde{A} = \lambda \det A$.

(e) If $\tilde{A}$ is obtained from $A$ by the operation $(E_i + \lambda E_j) \rightarrow (E_i)$ with $i \neq j$, then $\det \tilde{A} = \det A$.

(f) If $B$ is also an $n \times n$ matrix, then $\det AB = \det A \det B$.

(g) $\det A' = \det A$.

(h) When $A^{-1}$ exists, $\det A^{-1} = (\det A)^{-1}$.

(i) If $A$ is an upper triangular, or a lower triangular, or a diagonal matrix, then $\det A = \prod_{i=1}^{n} a_{ii}$.

**Theorem 6.16** The following statements are equivalent for any $n \times n$ matrix $A$:

(a) The equation $Ax = 0$ has the unique solution $x = 0$.

(b) The system $Ax = b$ has a unique solution for any $n$-dimensional column vector $b$.

(c) The matrix $A$ is nonsingular; that is, $A^{-1}$ exists.

(d) $\det A \neq 0$.

(e) Gaussian elimination with row interchanges can be performed on the system $Ax = b$ for any $n$-dimensional column vector $b$.

**Chapter 7:**

**Definition 7.1** A vector norm on $\mathbb{R}^n$ is a function, $\| \cdot \|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\| x \| \geq 0$ for all $x \in \mathbb{R}^n$,

(ii) $\| x \| = 0$ if and only if $x = 0$,

(iii) $\| \alpha x \| = |\alpha| \| x \|$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

(iv) $\| x + y \| \leq \| x \| + \| y \|$ for all $x, y \in \mathbb{R}^n$.

**Definition 7.2** The $l_2$ and $l_\infty$ norms for the vector $x = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$
\| x \|_2 = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}
$$

and

$$
\| x \|_\infty = \max_{1 \leq i \leq n} |x_i|.
$$

**Theorem 7.3** (Cauchy–Bunyakovsky–Schwarz Inequality for Sums)

For each $x = (x_1, x_2, \ldots, x_n)^t$ and $y = (y_1, y_2, \ldots, y_n)^t$ in $\mathbb{R}^n$,

$$
x'y = \sum_{i=1}^{n} x_i y_i \leq \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} y_i^2 \right)^{1/2} = \| x \|_2 \cdot \| y \|_2.
$$

**Definition 7.4** If $x = (x_1, x_2, \ldots, x_n)^t$ and $y = (y_1, y_2, \ldots, y_n)^t$ are vectors in $\mathbb{R}^n$, the $l_2$ and $l_\infty$ distances between $x$ and $y$ are defined by

$$
\| x - y \|_2 = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}
$$

and

$$
\| x - y \|_\infty = \max_{1 \leq i \leq n} |x_i - y_i|.
$$
**Definition 7.5** A sequence \( \{x^{(k)}\}_{k=1}^{\infty} \) of vectors in \( \mathbb{R}^n \) is said to converge to \( x \) with respect to the norm \( \| \cdot \| \) if, given any \( \varepsilon > 0 \), there exists an integer \( N(\varepsilon) \) such that

\[
\|x^{(k)} - x\| < \varepsilon, \quad \text{for all } k \geq N(\varepsilon).
\]

**Theorem 7.6** The sequence of vectors \( \{x^{(k)}\} \) converges to \( x \) in \( \mathbb{R}^n \) with respect to \( \| \cdot \|_\infty \) if and only if

\[
\lim_{k \to \infty} x^{(k)}_i = x_i, \quad \text{for each } i = 1, 2, \ldots, n.
\]

**Theorem 7.7** For each \( x \in \mathbb{R}^n \),

\[
\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty.
\]

**Definition 7.8** A matrix norm on the set of all \( n \times n \) matrices is a real-valued function, \( \| \cdot \| \), defined on this set, satisfying for all \( n \times n \) matrices \( A \) and \( B \) and all real numbers \( \alpha \):

(i) \( \|A\| \geq 0 \);

(ii) \( \|A\| = 0 \) if and only if \( A \) is \( 0 \), the matrix with all 0 entries;

(iii) \( \|\alpha A\| = |\alpha| \|A\| \);

(iv) \( \|A + B\| \leq \|A\| + \|B\| \);

(v) \( \|AB\| \leq \|A\| \|B\| \).

The distance between \( n \times n \) matrices \( A \) and \( B \) with respect to this matrix norm is \( \|A - B\| \).

**Theorem 7.9** If \( \| \cdot \| \) is a vector norm on \( \mathbb{R}^n \), then

\[
\|A\| = \max_{\|x\|=1} \|Ax\|
\]

is a matrix norm.

**Corollary 7.10** For any vector \( z \neq 0 \), matrix \( A \), and any natural norm \( \| \cdot \| \), we have

\[
\|Az\| \leq \|A\| \cdot \|z\|.
\]

**Theorem 7.11** If \( A = (a_{ij}) \) is an \( n \times n \) matrix, then

\[
\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.
\]

**Definition 7.12** If \( A \) is a square matrix, the **characteristic polynomial** of \( A \) is defined by

\[
p(\lambda) = \det(A - \lambda I).
\]

**Definition 7.13** If \( p \) is the characteristic polynomial of the matrix \( A \), the zeros of \( p \) are **eigenvalues**, or characteristic values, of the matrix \( A \). If \( \lambda \) is an eigenvalue of \( A \) and \( x \neq 0 \) satisfies \( (A - \lambda I)x = 0 \), then \( x \) is an **eigenvector**, or characteristic vector, of \( A \) corresponding to the eigenvalue \( \lambda \).

**Definition 7.14** The **spectral radius** \( \rho(A) \) of a matrix \( A \) is defined by

\[
\rho(A) = \max |\lambda|, \quad \text{where } \lambda \text{ is an eigenvalue of } A.
\]

(Recall that for complex \( \lambda = \alpha + \beta i \), we have \( |\lambda| = (\alpha^2 + \beta^2)^{1/2} \).)
Theorem 7.15 If $A$ is an $n \times n$ matrix, then

\begin{align*}
(i) \quad & \|A\|_2 = [\rho(A^*A)]^{1/2}, \\
(ii) \quad & \rho(A) \leq \|A\|, \text{ for any natural norm } \| \cdot \|.
\end{align*}

Definition 7.16 We call an $n \times n$ matrix $A$ convergent if

$$\lim_{k \to \infty} (A^k)_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, n \quad \text{and} \quad j = 1, 2, \ldots, n.$$ 

Theorem 7.17 The following statements are equivalent.

\begin{align*}
(i) \quad & A \text{ is a convergent matrix.} \\
(ii) \quad & \lim_{n \to \infty} \|A^n\| = 0, \text{ for some natural norm.} \\
(iii) \quad & \lim_{n \to \infty} \|A^n\| = 0, \text{ for all natural norms.} \\
(iv) \quad & \rho(A) < 1. \\
(v) \quad & \lim_{n \to \infty} A^n x = 0, \text{ for every } x.
\end{align*}

The method of Example 1 is called the Jacobi iterative method. It consists of solving the $i$th equation in $Ax = b$ for $x_i$ to obtain (provided $a_{ii} \neq 0$)

$$x_i = \sum_{j=1}^{n} \left( \frac{a_{ij}x_j}{a_{ii}} \right) + \frac{b_i}{a_{ii}}, \quad \text{for } i = 1, 2, \ldots, n$$

and generating each $x^{(k)}_i$ from components of $x^{(k-1)}$ for $k \geq 1$ by

$$x^{(k)}_i = \sum_{j=1}^{n} \left( \frac{-a_{ij}x^{(k-1)}_j}{a_{ii}} \right) + \frac{b_i}{a_{ii}}, \quad \text{for } i = 1, 2, \ldots, n.$$ \hspace{1cm} (7.4)

A possible improvement in Algorithm 7.1 can be seen by reconsidering Eq. (7.4). The components of $x^{(k-1)}$ are used to compute $x^{(k)}_i$. Since, for $i > 1$, $x^{(k)}_1, \ldots, x^{(k)}_{i-1}$ have already been computed and are probably better approximations to the actual solutions $x_1, \ldots, x_{i-1}$ than $x^{(k-1)}_1, \ldots, x^{(k-1)}_{i-1}$, it seems more reasonable to compute $x^{(k)}_i$ using these most recently calculated values. That is, we can use

$$x^{(k)}_i = -\sum_{j=1}^{i-1} (a_{ij}x^{(k)}_j) - \sum_{j=i+1}^{n} (a_{ij}x^{(k-1)}_j) + b_i,$$ \hspace{1cm} (7.7)

for each $i = 1, 2, \ldots, n$, instead of Eq. (7.4). This modification is called the Gauss-Seidel iterative technique and is illustrated in the following example.

Lemma 7.18 If the spectral radius $\rho(T)$ satisfies $\rho(T) < 1$ then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \ldots = \sum_{i=0}^{\infty} T^i.$$
Theorem 7.19  For any $x^{(0)} \in \mathbb{R}^n$, the sequence $\{x^{(k)}\}_{k=0}^\infty$ defined by
\[
x^{(k)} = T x^{(k-1)} + c, \quad \text{for each } k \geq 1,
\]
converges to the unique solution of $x = T x + c$ if and only if $\rho(T) < 1$. □

Corollary 7.20  If $\|T\| < 1$ for any natural matrix norm and $c$ is a given vector, then the sequence $\{x^{(k)}\}_{k=0}^\infty$ defined by $x^{(k)} = T x^{(k-1)} + c$ converges, for any $x^{(0)} \in \mathbb{R}^n$, to a vector $x \in \mathbb{R}^n$, and the following error bounds hold:

(i) $\|x - x^{(k)}\| \leq \|T\|^k \|x^{(0)} - x\|$

(ii) $\|x - x^{(k)}\| \leq \frac{\|T\|^k}{1 - \|T\|} \|x^{(1)} - x^{(0)}\|$. □