Chapter 5, Section 5.1

Definition 5.1 A function $f(t, y)$ is said to satisfy a **Lipschitz condition** in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2,)| \leq L|y_1 - y_2|,$$

whenever $(t, y_1), (t, y_2) \in D$. The constant L is called a **Lipschitz constant** for f. ■

Definition 5.2 A set $D \subset \mathbb{R}^2$ is said to be **convex** if whenever (t_1, y_1) and (t_2, y_2) belong to D and λ is in $[0, 1]$, the point $((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$ also belongs to D. ■

Theorem 5.3 Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq L, \quad \text{for all } (t, y) \in D, \tag{5.1}$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Theorem 5.4 Suppose that $D = \{(t, y) \mid a \leq t \leq b, -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \leq t \leq b$. ■

Definition 5.5 The initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha, \tag{5.2}$$

is said to be a **well-posed problem** if:

- A unique solution, $y(t)$, to the problem exists, and
- There exist constants $\varepsilon_0 > 0$ and $k > 0$ such that for any ε, with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all t in $[a, b]$, and when $|\delta_0| < \varepsilon$, the initial-value problem

$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \leq t \leq b, \quad z(a) = \alpha + \delta_0, \tag{5.3}$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a, b].$$ ■

Theorem 5.6 Suppose $D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

is well-posed. ■

Section 5.2

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \ldots, N$, by deleting the remainder term. Thus, Euler's method is

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1. \tag{5.8}$$

Equation (5.8) is called the **difference equation** associated with Euler's method

Theorem 5.9 Suppose f is continuous and satisfies a Lipschitz condition with constant L on

$$D = \{(t, y) \mid a \le t \le b, -\infty < y < \infty\}$$

and that a constant M exists with

$$|y''(t)| \le M, \quad \text{for all } t \in [a, b].$$

Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

and $w_0, w_1, \ldots, w_N$ be the approximations generated by Euler's method for some positive integer N. Then, for each $i = 0, 1, 2, \ldots, N$,

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right]. \tag{5.10}$$

Theorem 5.10 Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha \tag{5.12}$$

and $u_0, u_1, \ldots, u_N$ be the approximations obtained using Eq. (5.11). If $|\delta_i| < \delta$ for each $i = 0, 1, \ldots, N$ and the hypotheses of Theorem 5.9 hold for Eq. (5.12), then

$$|y(t_i) - u_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)[e^{L(t_i - a)} - 1] + |\delta_0|e^{L(t_i - a)}, \tag{5.13}$$

for each $i = 0, 1, \ldots, N$. ■

Section 5.3

Definition 5.11 The difference method

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1,$$

has **local truncation error**

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \ldots, N - 1$.

Taylor method of order n:

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1, \tag{5.17}$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!}f^{(n-1)}(t_i, w_i).$$

Note that Euler's method is Taylor's method of order one.

Theorem 5.12 If Taylor's method of order n is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad a \le t \le b, \quad y(a) = \alpha,$$

with step size h and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$. ■

Section 5.4

Theorem 5.13 Suppose that $f(t, y)$ and all its partial derivatives of order less than or equal to $n + 1$ are continuous on $D = \{(t, y) \mid a \le t \le b, c \le y \le d\}$, and let $(t_0, y_0) \in D$. For every $(t, y) \in D$, there exists ξ between t and t_0 and μ between y and y_0 with

Taylor's Theorem in two variables

$$f(t, y) = P_n(t, y) + R_n(t, y),$$

where

$$\begin{aligned}
P_n(t, y) = f(t_0, y_0) &+ \left[(t - t_0)\frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(t_0, y_0)\right] \\
&+ \left[\frac{(t - t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0, y_0)\right. \\
&+ \left.\frac{(y - y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0, y_0)\right] + \cdots \\
&+ \left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t - t_0)^{n-j}(y - y_0)^j \frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0, y_0)\right]
\end{aligned}$$

and

$$R_n(t, y) = \frac{1}{(n + 1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t - t_0)^{n+1-j}(y - y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(\xi, \mu).$$

The function $P_n(t, y)$ is called the **nth Taylor polynomial in two variables** for the function f about (t_0, y_0), and $R_n(t, y)$ is the remainder term associated with $P_n(t, y)$. ■

The difference-equation method resulting from replacing $T^{(2)}(t, y)$ in Taylor's method of order two by $f(t + (h/2), y + (h/2)f(t, y))$ is a specific Runge–Kutta method known as the *Midpoint method*.

Midpoint Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right), \quad \text{for each } i = 0, 1, \ldots, N - 1.$$

Two other $O(h^2)$ methods of the form: $a_1 f(t, y) + a_2 f(t + \alpha_2, y + \delta_2 f(t, y))$ — [5.21]

Modified Euler Method

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))],$$
$$\text{for each} \quad i = 0, 1, 2, \ldots, N-1.$$

Heun's Method

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + \frac{h}{4}\left[f(t_i, w_i) + 3f\left(t_i + \frac{2}{3}h, w_i + \frac{2}{3}hf(t_i, w_i)\right)\right],$$
$$\text{for each } i = 0, 1, 2, \ldots, N-1.$$

Runge–Kutta Order Four:

$$w_0 = \alpha,$$
$$k_1 = hf(t_i, w_i),$$
$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right),$$
$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right),$$
$$k_4 = hf(t_{i+1}, w_i + k_3),$$
$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

for each $i = 0, 1, \ldots, N-1$. This method has local truncation error $O(h^4)$, provided the solution $y(t)$ has five continuous derivatives.

Chapter 6:

Theorem 6.5 Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(a) $A + B = B + A$, **(b)** $(A + B) + C = A + (B + C)$,

(c) $A + O = O + A = A$, **(d)** $A + (-A) = -A + A = 0$,

(e) $\lambda(A + B) = \lambda A + \lambda B$, **(f)** $(\lambda + \mu)A = \lambda A + \mu A$,

(g) $\lambda(\mu A) = (\lambda\mu)A$, **(h)** $1A = A$.

Theorem 6.9 Let A be an $n \times m$ matrix, B be an $m \times k$ matrix, C be a $k \times p$ matrix, D be an $m \times k$ matrix, and λ be a real number. The following properties hold:

(a) $A(BC) = (AB)C$; (Associative law for multiplication)

(b) $A(B + D) = AB + AD$; (Distributive law)

(c) $I_m B = B$ and $B I_k = B$; (Identity Element)

(d) $\lambda(AB) = (\lambda A)B = A(\lambda B)$. ■

square matrix

Definition 6.10 An $n \times n$ matrix A is **nonsingular** (or *invertible*) if an $n \times n$ matrix A^{-1} exists with $AA^{-1} = A^{-1}A = I$. The matrix A^{-1} is called the **inverse** of A. A matrix without an inverse is called **singular** (or *noninvertible*). ■

The following properties regarding matrix inverses follow from Definition 6.10. The proofs of these results are considered in Exercise 5.

Theorem 6.11 For any nonsingular $n \times n$ matrix A:

(a) A^{-1} is unique.

(b) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

(c) If B is also a nonsingular $n \times n$ matrix, then $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 6.13 The following operations involving the transpose of a matrix hold whenever the operation is possible:

(a) $(A^t)^t = A$, (b) $(A + B)^t = A^t + B^t$,

(c) $(AB)^t = B^t A^t$, (d) if A^{-1} exists, then $(A^{-1})^t = (A^t)^{-1}$.

Definition 6.14

(a) If $A = [a]$ is a 1×1 matrix, then $\det A = a$.

(b) If A is an $n \times n$ matrix, the **minor** M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of the matrix A.

(c) The **cofactor** A_{ij} associated with M_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.

(d) The **determinant** of the $n \times n$ matrix A, when $n > 1$, is given either by

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } i = 1, 2, \ldots, n,$$

or by

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } j = 1, 2, \ldots, n.$$ ■

Theorem 6.15 Suppose A is an $n \times n$ matrix:

(a) If any row or column of A has only zero entries, then $\det A = 0$.

(b) If A has two rows or two columns the same, then $\det A = 0$.

elementary row operations

(c) If $\tilde{A}$ is obtained from A by the operation $(E_i) \leftrightarrow (E_j)$, with $i \neq j$, then $\det \tilde{A} = -\det A$.

(d) If $\tilde{A}$ is obtained from A by the operation $(\lambda E_i) \rightarrow (E_i)$, then $\det \tilde{A} = \lambda \det A$.

(e) If $\tilde{A}$ is obtained from A by the operation $(E_i + \lambda E_j) \rightarrow (E_i)$ with $i \neq j$, then $\det \tilde{A} = \det A$.

(f) If B is also an $n \times n$ matrix, then $\det AB = \det A \det B$.

(g) $\det A^t = \det A$.

(h) When A^{-1} exists, $\det A^{-1} = (\det A)^{-1}$.

(i) If A is an upper triangular, or a lower triangular, or a diagonal matrix, then $\det A = \prod_{i=1}^{n} a_{ii}$. ∎

Theorem 6.16 The following statements are equivalent for any $n \times n$ matrix A:

(a) The equation $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.

(b) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any n-dimensional column vector $\mathbf{b}$.

(c) The matrix A is nonsingular; that is, A^{-1} exists.

(d) $\det A \neq 0$.

(e) Gaussian elimination with row interchanges can be performed on the system $A\mathbf{x} = \mathbf{b}$ for any n-dimensional column vector $\mathbf{b}$. ∎

Chapter 7:

Definition 7.1 A **vector norm** on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. ∎

$\|x\|$ defines a measure for the magnitude of vector x

Definition 7.2 The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$\|\mathbf{x}\|_2 = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Theorem 7.3 **(Cauchy–Bunyakovsky–Schwarz Inequality for Sums)**

For each $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ in $\mathbb{R}^n$,

$$\mathbf{x}^t\mathbf{y} = \sum_{i=1}^{n} x_i y_i \leq \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{1/2} \left\{ \sum_{i=1}^{n} y_i^2 \right\}^{1/2} = \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2.$$

Definition 7.4 If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ are vectors in $\mathbb{R}^n$, the l_2 and l_∞ distances between $\mathbf{x}$ and $\mathbf{y}$ are defined by

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{1/2} \quad \text{and} \quad \|\mathbf{x} - \mathbf{y}\|_\infty = \max_{1 \leq i \leq n} |x_i - y_i|. \quad ∎$$

Definition 7.5 A sequence $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to **converge** to $\mathbf{x}$ with respect to the norm $\|\cdot\|$ if, given any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| < \varepsilon, \quad \text{for all } k \geq N(\varepsilon). \quad \blacksquare$$

Theorem 7.6 The sequence of vectors $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in $\mathbb{R}^n$ with respect to $\|\cdot\|_\infty$ if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$, for each $i = 1, 2, \ldots, n$. ∎

Theorem 7.7 For each $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty.$$

Definition 7.8 A **matrix norm** on the set of all $n \times n$ matrices is a real-valued function, $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

(i) $\|A\| \geq 0$;

(ii) $\|A\| = 0$, if and only if A is O, the matrix with all 0 entries;

(iii) $\|\alpha A\| = |\alpha|\|A\|$;

(iv) $\|A + B\| \leq \|A\| + \|B\|$;

(v) $\|AB\| \leq \|A\|\|B\|$. ∎

The **distance between $n \times n$ matrices** A and B with respect to this matrix norm is $\|A - B\|$.

Theorem 7.9 If $||\cdot||$ is a vector norm on $\mathbb{R}^n$, then

$$\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

is a matrix norm.

Corollary 7.10 For any vector $\mathbf{z} \neq \mathbf{0}$, matrix A, and any natural norm $\|\cdot\|$, we have

$$\|A\mathbf{z}\| \leq \|A\| \cdot \|\mathbf{z}\|.$$

Theorem 7.11 If $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|.$$

Definition 7.12 If A is a square matrix, the **characteristic polynomial** of A is defined by

$$p(\lambda) = \det(A - \lambda I).$$

Definition 7.13 If p is the characteristic polynomial of the matrix A, the zeros of p are **eigenvalues**, or characteristic values, of the matrix A. If λ is an eigenvalue of A and $\mathbf{x} \neq \mathbf{0}$ satisfies $(A - \lambda I)\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is an **eigenvector**, or characteristic vector, of A corresponding to the eigenvalue λ. ∎

Definition 7.14 The **spectral radius** $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max|\lambda|, \quad \text{where } \lambda \text{ is an eigenvalue of } A.$$

(Recall that for complex $\lambda = \alpha + \beta i$, we have $|\lambda| = (\alpha^2 + \beta^2)^{1/2}$.)

Theorem 7.15 If A is an $n \times n$ matrix, then

(i) $\|A\|_2 = [\rho(A^t A)]^{1/2}$,

(ii) $\rho(A) \le \|A\|$, for any natural norm $\|\cdot\|$.

Definition 7.16 We call an $n \times n$ matrix A **convergent** if

$$\lim_{k\to\infty} (A^k)_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, n \quad \text{and} \quad j = 1, 2, \ldots, n.$$

Theorem 7.17 The following statements are equivalent.

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

(iii) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

(iv) $\rho(A) < 1$.

(v) $\lim_{n\to\infty} A^n\mathbf{x} = \mathbf{0}$, for every $\mathbf{x}$.

The method of Example 1 is called the **Jacobi iterative method**. It consists of solving the ith equation in $A\mathbf{x} = \mathbf{b}$ for x_i to obtain (provided $a_{ii} \neq 0$)

$$x_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \left(-\frac{a_{ij}x_j}{a_{ii}}\right) + \frac{b_i}{a_{ii}}, \qquad \text{for } i = 1, 2, \ldots, n$$

and generating each $x_i^{(k)}$ from components of $\mathbf{x}^{(k-1)}$ for $k \ge 1$ by

$$x_i^{(k)} = \frac{\sum_{\substack{j=1\\ j\neq i}}^{n} \left(-a_{ij}x_j^{(k-1)}\right) + b_i}{a_{ii}}, \qquad \text{for } i = 1, 2, \ldots, n. \tag{7.4}$$

A possible improvement in Algorithm 7.1 can be seen by reconsidering Eq. (7.4). The components of $\mathbf{x}^{(k-1)}$ are used to compute $x_i^{(k)}$. Since, for $i > 1$, $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$ have already been computed and are probably better approximations to the actual solutions $x_1, \ldots, x_{i-1}$ than $x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}$, it seems more reasonable to compute $x_i^{(k)}$ using these most recently calculated values. That is, we can use

$$x_i^{(k)} = \frac{-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)}) + b_i}{a_{ii}}, \tag{7.7}$$

for each $i = 1, 2, \ldots, n$, instead of Eq. (7.4). This modification is called the **Gauss–Seidel iterative technique** and is illustrated in the following example.

Lemma 7.18 If the spectral radius $\rho(T)$ satisfies $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j.$$

Theorem 7.19 For any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

(Proof skipped)

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad \text{for each } k \geq 1, \tag{7.10}$$

converges to the unique solution of $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ if and only if $\rho(T) < 1$. ■

Corollary 7.20 If $\|T\| < 1$ for any natural matrix norm and $\mathbf{c}$ is a given vector, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ converges, for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, to a vector $\mathbf{x} \in \mathbb{R}^n$, and the following error bounds hold:

(i) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$;

(ii) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \dfrac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$. ■