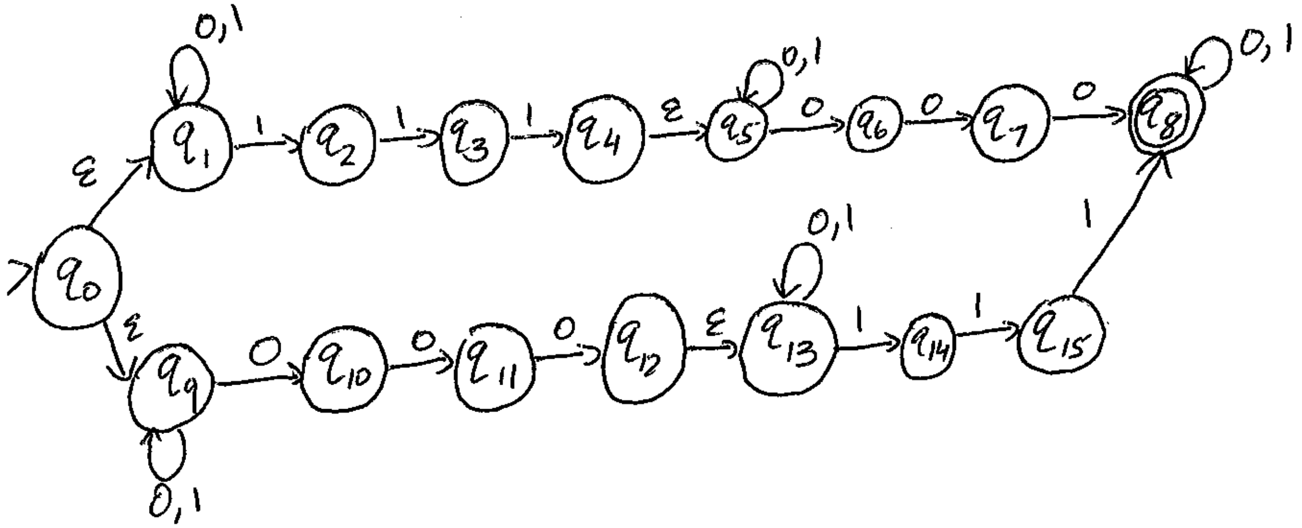
COT 4210: Discrete Structures II
Final Exam Solution
April 30, 2013

Lecturer: Arup Guha

(Directions: Please justify your answer to each question. No answer, even if it is correct, will be given full credit without the proper justification.)

1) (10 pts) Design an NFA that accepts all strings over the alphabet {0,1} that contain the substring 111 and the substring 000. Make sure to use features unique to NFAs that are not allowed in DFAs in your design. Please clearly label your start state, your final state(s) and each transition. Please name your states q_0, q_1, etc. with q_0 being the start state.



Grading

1pt label start state

1pt use of non-determinism

1pt label accept state(s)

2 pts accepts all strings of form $(0 \cup 1)^* 000 (0 \cup 1)^* 111 (0 \cup 1)^*$

2 pts accepts all strings of form $(0 \cup 1)^* 111 (0 \cup 1)^* 000 (0 \cup 1)^*$

3pts doesn't accept any extra strings incorrectly.

2) (10 pts) Convert the following Context Free Grammar below over the alphabet {0, 1} to Chomsky Normal Form. Clearly label each of the four steps in the algorithm and the resulting grammar after each of the steps.

$S \rightarrow AAB \mid 0 \mid 11$
$A \rightarrow 1 \mid BBBB$
$B \rightarrow A \mid BA \mid 10 \mid \varepsilon$

1. $S_0 \rightarrow S$ (1 pt)

2. Remove $B \rightarrow \varepsilon$
Add:
$A \rightarrow BBB \mid BB \mid B \mid \varepsilon$
$S \rightarrow AA$

Remove $A \rightarrow \varepsilon$
Add:
$S \rightarrow AB \mid B \mid A \mid \varepsilon$

Remove $S \rightarrow \varepsilon$
Add: $S_0 \rightarrow \varepsilon$

Grammar after step 2 (3pts)

$S_0 \rightarrow S \mid \varepsilon$
$S \rightarrow AAB \mid 0 \mid 11 \mid AA \mid AB \mid B \mid A$
$A \rightarrow 1 \mid BBBB \mid BBB \mid BB \mid B$
$B \rightarrow A \mid BA \mid 10$

3. Remove $B \rightarrow A$:
Add:
$B \rightarrow 1 \mid BBBB \mid BBB \mid BB$

Remove $A \rightarrow B$:
Add
$A \rightarrow BA \mid 10$

(4pts)

Remove $S \rightarrow A$, $S \rightarrow B$
Add:
$S \rightarrow BBBB \mid BBB \mid BB \mid 1 \mid BA \mid 10$

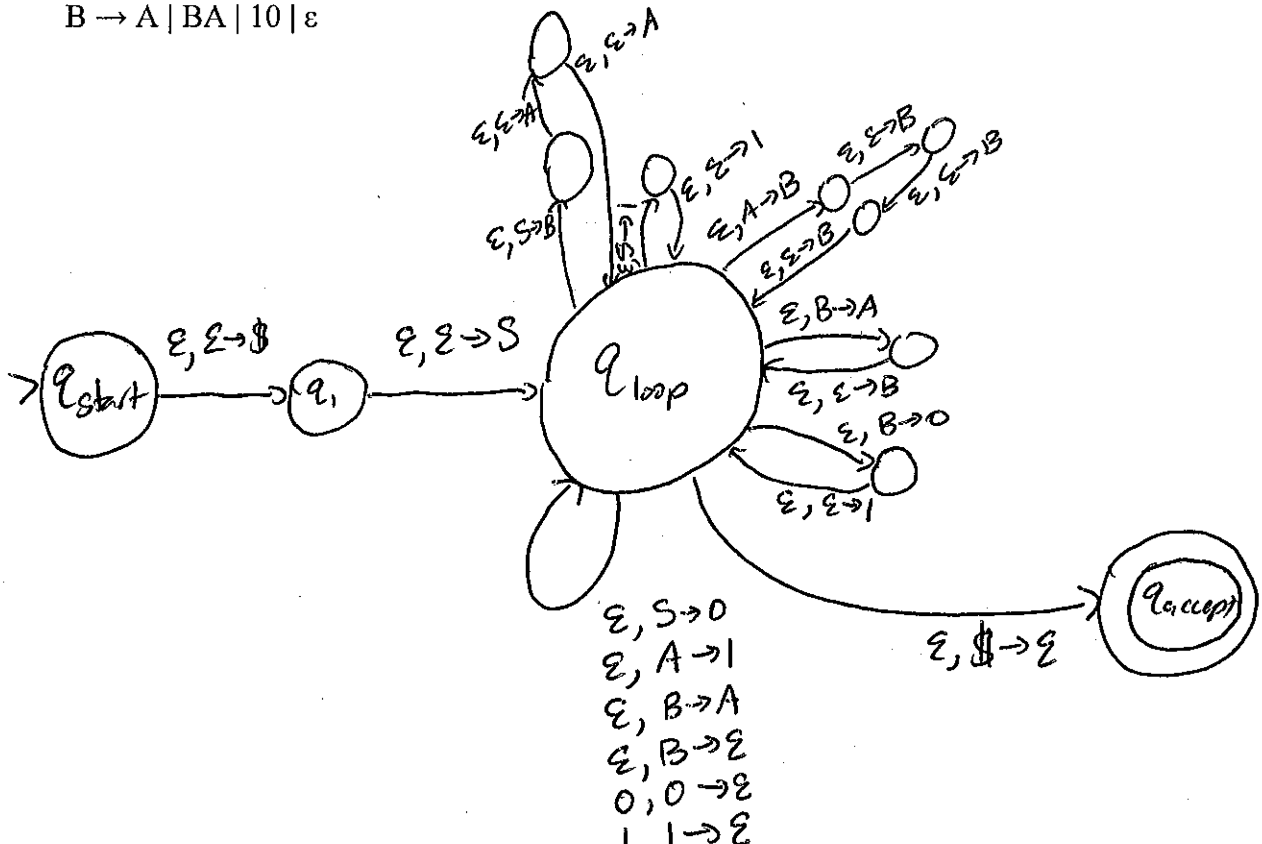
Remove $S_0 \rightarrow S$
Add
$S_0 \rightarrow AAB \mid 0 \mid 11 \mid AA \mid AB \mid BBBB \mid BBB \mid BB \mid 1 \mid BA \mid 10$

4. Add additional variables: (2pts)

$C \rightarrow 0$, $D \rightarrow 1$
$E \rightarrow BB$, $F \rightarrow AB$
$S_0 \rightarrow AF \mid 0 \mid DD \mid AA \mid AB \mid EE \mid EB \mid BB \mid 1 \mid BA \mid DC \mid \varepsilon$
$S \rightarrow AF \mid 0 \mid DD \mid AA \mid AB \mid EE \mid EB \mid BB \mid 1 \mid BA \mid DC$
$A \rightarrow 1 \mid EE \mid EB \mid BB \mid BA \mid DC$
$B \rightarrow BA \mid DC \mid 1 \mid EE \mid EB \mid BB$

3) (10 pts) Using the algorithm in the text, convert the Context Free Grammar in question #2 (which is replicated below for convenience) to an equivalent PDA that accepts the exact same language described by the Context Free Grammar.

$S \rightarrow AAB \mid 0 \mid 11$
$A \rightarrow 1 \mid BBBB$
$B \rightarrow A \mid BA \mid 10 \mid \varepsilon$



Grading

q_{start} - 1 pt
Push $, S - 1 pt
q_{loop} - 1 pt
"loops" for each multi-rule - 2 pts
single rule q_{loop} - 1 pt
rules 0,0→ε ; 1,1→ε - 1 pt
q_{accept} transition - 1 pt
ORDER vars in multi-rules - 2 pts
(in reverse order)

4) (10 pts) Let L = { <M, S> | M is a standard TM and S is a finite set of strings over {0,1} such that there exists some string t ∈ S^*, such that M accepts t.} Prove that L is undecidable without Rice's Theorem.

Assume to the contrary, that L is decidable. (1 pt)
Let TM M' decide membership in L. (1 pt)
We will design TM R to decide A_{TM} using M' as follows: (1 pt)

R(TM M, String w) {

(5pts) Construct TM M_1 such that if its input isn't w, it rejects. If its input is w, it runs the steps of M.
Thus, $L(M_1) = \emptyset$ if M doesn't accept w and
$L(M_1) = \{w\}$ if M does accept w.

Return $M'(M_1, \{w\})$

}

(2pts) If R accepts, then M' accepted. If M' accepts then a string of the form $w^* \in L(M_1)$. __BUT__ $L(M_1) = \{w\}$ or $L(M_1) = \emptyset$. It follows that if a string of the form $w^* \in L(M_1)$, then $w \in L(M_1)$, which means M accepts w, as desired. Finally, when M' rejects, we know $w \notin L(M_1)$ so M doesn't accept w in that case. This proves the correctness of R.

Since A_{TM} is undecidable though, we can conclude that no decider to L exists. Hence, L is undecidable, as desired.

5) (10 pts) In the following solitaire game, you are given an m x m board. On each of its m^2 positions lies either a blue stone, a red stone or nothing at all. You play by removing stones from the board until each column contains only stones of a single color and each row contains at least one stone. You win if you achieve this objective. Winning may or may not be possible based on the initial configuration. Let SOLITAIRE = { <G> | G is a winnable solitaire game configuration }. Prove that SOLITAIRE is NP-Complete by reducing 3-CNF-SAT to it.

0pts

First, note that SOLITAIRE $\in$ NP since given a certificate for the color that remains in each column, we can remove all illegal stones and check if each row has at least 1 stone left, in polynomial time of the input size.

We map our 3-CNF-SAT input of n variables and C clauses to a rectangle, B, of C rows and n columns as follows:

⓪ Remove any clauses that contain $x_i \vee \overline{x_i}$.

① Label each row with a clause number, 1 through C.

② Label each column with a variable number, 1 through n.

6pts

③ For each (i,j) such that C_i contains x_j, place a red stone in B[i][j]. For each (i,j) such that C_i contains $\overline{x_j}$ place a blue stone.

2pts

Iff the 3-CNF-SAT instance is satisfiable can we win this solitaire game. If an instance is satisfiable, take a valid truth setting and leave only these corresponding stones on the board. For example, if x_3 = false, leave only blue stones in column 3. Each row with one of these stones represents a satisfied clause with x_3 = false. Similarly, any solution to the solitaire instance can be used to find a valid truth setting in the original formula by taking the end board, finding the color in each column and setting the corresponding variable to true if the color is red, and false if it's blue.

2pts

Finally, to output a square board, add n rows to the bottom

8) (2 pts) Name one ingredient in Raisinets. Raisins (2pts for all)

0pts

and C columns to the right. Put no stones in B[i][j] for all $i \le C$, $j \ge n+1$. Put no stones in B[i][j] for all $i \ge C+1$, $j \le n$ or $j \ge n+2$. Place 1 Red Stone in B[i][n+1] for all $i \ge C+1$. Any solution to the original board will map to a solution on this square board by selecting red for column n+1,