# WEEK #3



## 1. REGULAR EXPRESSIONS

PRIMITIVE REG. EXPR. OVER $\Sigma$

| EXPRESSION | SET |
|---|---|
| $\emptyset$ | $\emptyset$ |
| $\lambda$ | $\{\lambda\}$ |
| $a,\ a \in \Sigma$ | $\{a\}$ |

CLOSURE OF EXPRESSIONS R, S DENOTING SETS R, S

| | |
|---|---|
| $R+S$ | $R \cup S = \{x \mid x \in R \text{ or } x \in S\}$ |
| $R \cdot S$ | $R \cdot S = \{xy \mid x \in R \text{ and } y \in S\}$ |
| $R^*$ | $R^* = \{\lambda\} \cup R \cup R^2 \cup \cdots$ |

FOR CONVENIENCE WE ALSO ALLOW

| | |
|---|---|
| $R^+$ | $R^+ = R \cup R^2 \cup R^3 \cup \cdots$ |

PRECEDENCE

$* > \cdot > +$

$\cdot$ AND $+$ ARE ASSOCIATED LEFT TO RIGHT

$*$ IS NON-ASSOCIATIVE OR JUST REALIZE $(R^*)^* = R^*$

PARENTHESES ARE USED TO OVERRIDE DEFAULT PARSE OR FOR READABILITY

## 2. REGULAR SETS

EVERY REGULAR EXPRESSION DENOTES A REGULAR SET

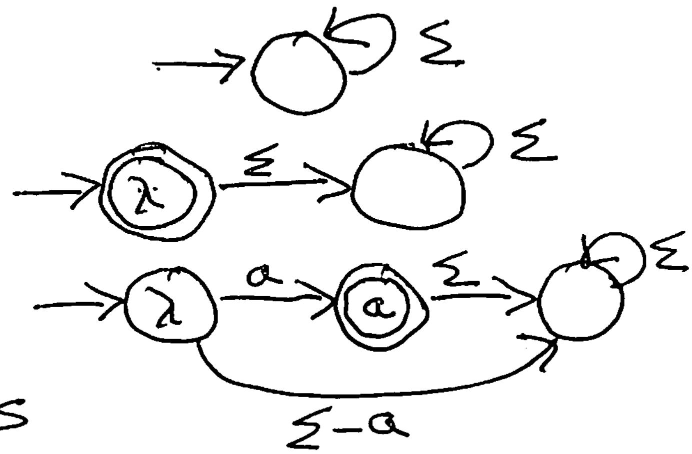
2. EVERY REGULAR SET IS A REGULAR LANG.

REGULAR EXPRESSION | DFA

$\emptyset$

$\lambda$

$a$



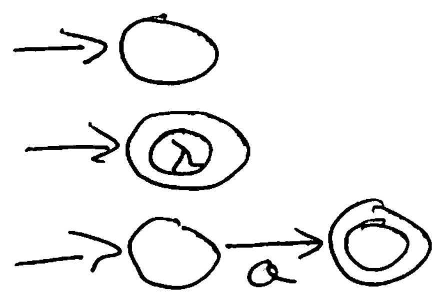

OR COULD DO NFA'S

REG EXP | NFA

$\emptyset$

$\lambda$

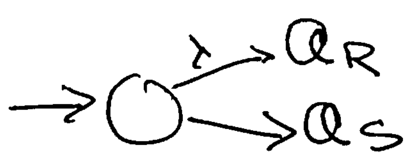
$a$



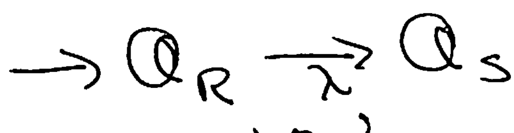

As REGULAR LANGUAGES CLOSED UNDER $\cup, \circ, *$

THEN GET ALL REG. SETS AS REG. LANGUAGES

$R+S$

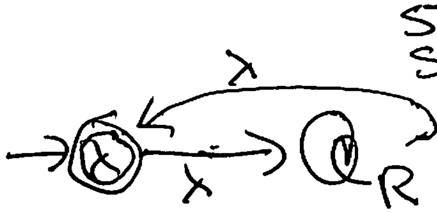



$R \circ S$




FROM EACH FINAL STATE OF $Q_R$ TO START STATE OF $Q_S$

$R^*$



# WEEK #3

## 3. EVERY REGULAR LANGUAGES IS A REG. SET

### a. APPROACH #1: $R_{ij}^{k}$ SETS (REG. EXPR.)

LET $L = \mathcal{L}(Q)$ WHERE

$Q = (Q, \Sigma, \delta, q_1, F)$, $Q = \{q_1, \ldots, q_n\}$

WANT TO BUILD EXPRESSIONS

$R_{ij}^{k}$, WHERE $1 \le i, j \le n$; $0 \le k \le n$

SEMANTICALLY,

$R_{ij}^{k} = \{ w \mid \delta^*(q_i, w) = q_j$ AND PATH FROM $q_i$ TO $q_j$ INVOLVES NO STATE WITH INDEX $> k \}$;

ACTUALLY $R_{ij}^{k}$ IS REG. EXPR FOR THIS SET

[EXAMPLE ON 8c/8d]

BASIS: $R_{ij}^{0} = \emptyset$ IF $i \ne j$ AND $\delta(q_i, a) \ne q_j \; \forall a \in \Sigma$

$R_{ij}^{0} = a$ IF $\delta(q_i, a) = q_j$ AND $i \ne j$

$R_{ij}^{0} = \lambda$ IF $i = j$ AND $\delta(q_i, a) \ne q_j \; \forall a \in \Sigma$

$R_{ij}^{0} = \lambda + a$ IF $i = j$ AND $\delta(q_i, a) = q_i$

INDUCTIVE HYP: ASSUME $R_{ij}^{m}$ REG EXPR (SET) $0 \le m \le k$, $1 \le i, j \le n$

INDUCTIVE STEP:

$$R_{ij}^{k+1} = R_{ij}^{k} + R_{i\,k+1}^{k} (R_{k+1,k+1}^{k})^{*} R_{k+1,j}^{k}$$

$$\mathcal{L}(Q) = +_{q_f \in F} R_{1f}^{n}$$

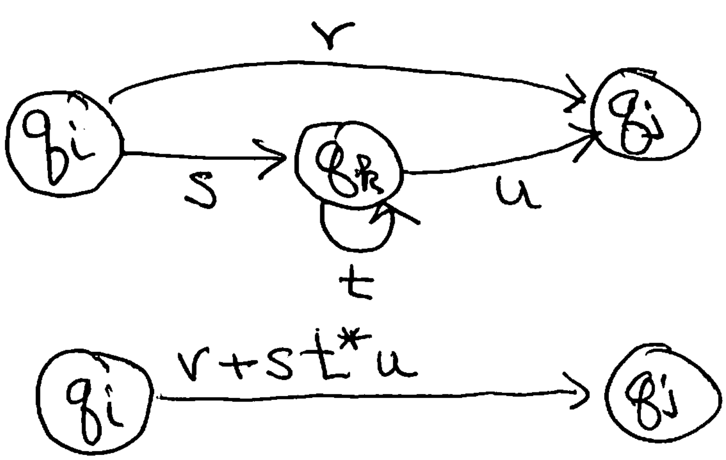
## 3. b. STATE RIPPING

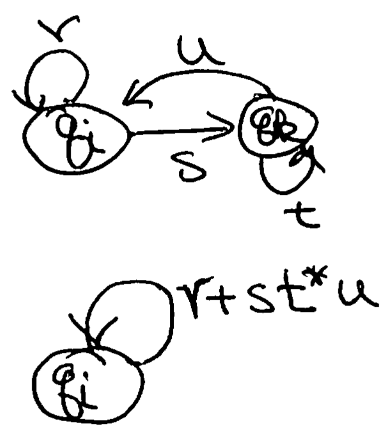
LET $L = \mathcal{L}(\mathcal{Q})$ WHERE
$\mathcal{Q} = (Q, \Sigma, \delta, q_1, F)$, $Q = \{q_1, \ldots, q_n\}$

CONSTRUCT
$\mathcal{Q}' = (Q \cup \{q_0, q_f\}, \Sigma, \delta', q_0, \{q_f\})$ $q_0, q_f \notin Q$
$\delta'(q_0, \lambda) = \{q_1\}$ $\delta'(q_j, \lambda) \supseteq \{q_f\}$ $\forall q_j \in F$
$r_{ij}$ INITIALLY IS REG. EXP THAT DIRECTLY GOES FROM $q_i$ TO $q_j$
WE WILL COLLAPSE STATES IN $\mathcal{Q}'$, EXCEPT FOR $q_0$ + $q_f$. AS WE RIP A STATE $q_k$ OUT, WE WILL EXTEND EXPRESSION $r_{ij}$ BY ARCS THAT LED INTO $q_k$ FROM $q_i$ AND, TRANSITIVELY TO $q_j$, BY AN EXPR. THAT REPRESENTS $r_{ik} r_{kk}^* r_{kj} + r_{ij}$

IN END $\mathcal{L}(\mathcal{Q}) = r_{0f}$

[EXAMPLE ON $q_2$-$q_J$]

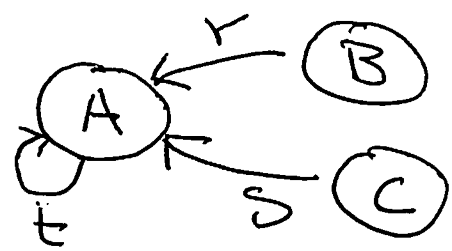



OF COURSE MUST DO FOR ALL PAIRS

# 3.C. REGULAR EQUATIONS

IF




THEN $A = Br + Cs + At$

IF A IS START STATE THEN

$$A = \lambda + Br + Cs + At$$

OF COURSE, THERE IS A TERM FOR EVERY STATE THAT HAS AN ARC LEADING INTO A.

DO THIS FOR ALL STATES AND HAVE $n$ EQUATIONS IN $n$ UNKNOWNS THAT SEEMS EASY BUT SOME MAY BE RECURSIVE IF AUTOMATA HAS LOOPS

WE CLAIM THAT IF $\lambda \notin P$ THEN

$$R = Q + RP$$

HAS UNIQUE SOLUTION

$$R = QP^*$$

[EXAMPLES OF 98/99]

3 c. CONTINUED

(1) $QP^*$ IS <u>A</u> SOLUTION

$$R = Q + RP = Q + QP^*P = Q + QP^+ = QP^*$$

(2) ANY SOLUTION IS CONTAINED IN $QP^*$

$$R = Q + RP = Q + (Q+RP)P = Q + QP + RP^2$$
$$= Q + QP + (Q+RP)P^2 = Q + QP + QP^2 + RP^3$$
$$\cdots$$
$$= Q + QP + QP^2 + QP^3 + \cdots + QP^k + RP^{k+1}, \; k \geq 0$$
$$= Q(\lambda + P + P^2 + \cdots + P^k) + RP^{k+1}, \; k \geq 0$$

SINCE $\lambda \notin P$ THEN FOR ANY $w \in R$, $|w| = k$, $w \notin RP^{k+1}$ AND SO

$$w \in Q(\lambda + P + P^2 + \cdots + P^k) \subseteq QP^*$$

THUS $R \subseteq QP^*$

(3) AS $QP^* \subseteq R$ AND $R \subseteq QP^*$

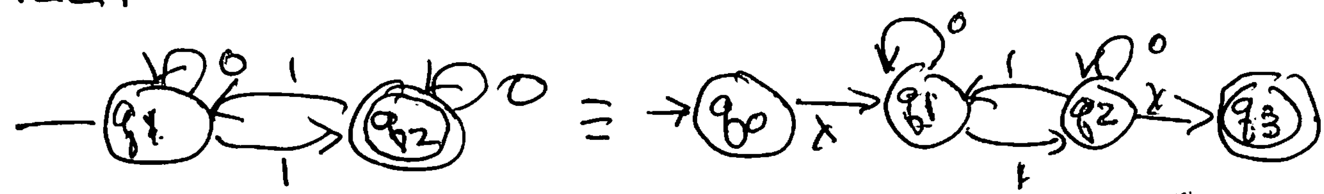
THEN $R = QP^*$

AGAIN PROVIDED $\lambda \notin P$

NOTE: WE WOULD ONLY SEE $\lambda \in P$ IF WE HAD


WHICH IS USELESS

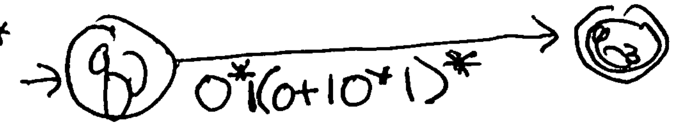
4. REALLY SIMPLE EXAMPLE



$R_{11}^{0} = \lambda + 0 \quad R_{22}^{0} = 0$

$R_{12}^{0} = 1 \quad R_{21}^{0} = 1$

$R_{11}^{1} = (\lambda+0) + (\lambda+0)(\lambda+0)^*(\lambda+0) = 0^*$



$R_{12}^{1} = 1 + (\lambda+0)(\lambda+0)^* 1 = 0^* 1$

$R_{22}^{1} = 0 + 1(\lambda+0)^* 1 = 0 + 10^* 1$

$R_{21}^{1} = 1 + 1(\lambda+0)^*(\lambda+0) = 10^*$

$R_{12}^{2} = 0^*1 + 0^*1(0+10^*1)^*(0+10^*1)$
$= 0^*1(0+10^*1)^*$

$Q_1 = \lambda + Q_2 1 + Q_1 0$

$Q_2 = Q_1 1 + Q_2 0$

$Q_1 = (\lambda + Q_2 1)0^*$

$Q_2 = (\lambda + Q_2 1)0^* 1 + Q_2 0$
$= 0^*1 + Q_2(0+10^*1)$
$= 0^*1(0+10^*1)^*$

VERY RARE THAT ALL THREE REG EXPR IN SAME FORM. NOTES HAVE BETTER EXAMPLE.

## 5. STATES AS EQUIV. CLASSES

NOTION OF STATES AS EQUIVALENCE CLASSES

LET $\mathcal{Q}=(Q,\Sigma,\delta,q_0,F)$ BE SOME DFA

ASSUME EVERY STATE IN $Q$ IS REACHABLE,

I.E., $\forall q \in Q \; \exists w \in \Sigma^* \ni \delta^*(q_0,w)=q$

CLEARLY, IF $\delta^*(q_0,x)=\delta^*(q_0,y)$

THEN $\forall z \; \delta^*(q_0,xz)=\delta^*(q_0,yz)$

DEFINE $R_{\mathcal{Q}}$ BY

$$x R_{\mathcal{Q}} y \text{ IFF } \delta^*(q_0,x)=\delta^*(q_0,y)$$

THEN $R_{\mathcal{Q}}$ IS AN EQUIV. REL. THAT PARTITIONS $\Sigma^*$ INTO $|Q|$ EQUIV. CLASSES AND THE UNION OF SOME OF THESE CLASSES DEFINES $\mathcal{L}(\mathcal{Q})$ -- CLASSES ASSOC. WITH $F$

6. MINIMIZATION OF DFAs

DEFINE STATES $p,q \in Q$ AS INDISTINGUISHABLE IF $\forall z\, [\delta^*(p,z) \in F \Leftrightarrow \delta^*(q,z) \in F]$

IF TWO STATES ARE INDISTINGUISHABLE THEN THEY HAVE SAME FUTURES AND CAN BE MERGED.

MERGING STATES CREATES A NEW, LESS REFINED, EQUIV. REL. OVER $\Sigma^*$ THAT STILL HAS PROPERTY THAT $\mathcal{L}(Q)$ IS UNION OF SOME CLASSES, BUT NUMBER OF CLASSES IS SMALLER.

NEW FINAL STATES ARE ASSOCIATED WITH EQUIV. CLASSES CONTAINING STRINGS THAT LED TO FINAL STATES IN Q.

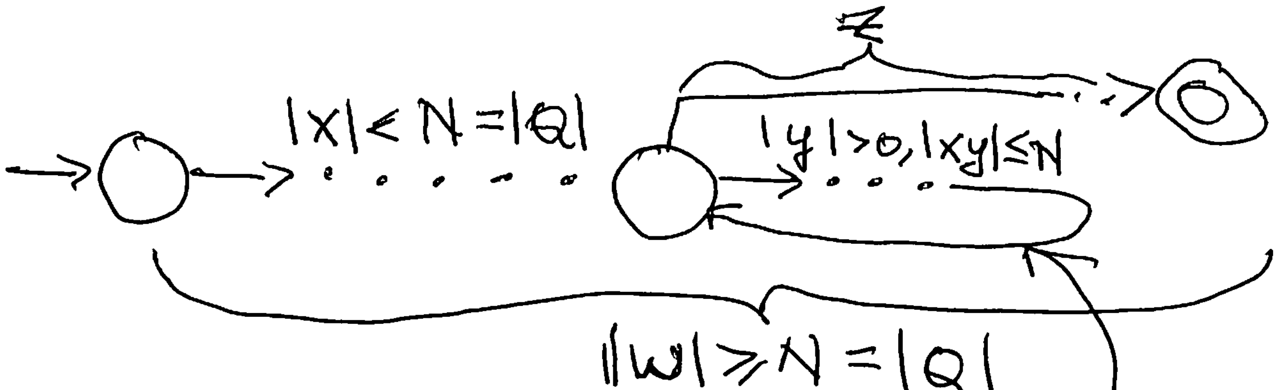
WE ACTUALLY ATTACK MINIMIZATION BY DISCOVERING DISTINGUISHABLE STATES

THE EASIEST APPROACH IS VIA A LOWER TRIANGULAR MATRIX

[GO TO 110/111]

# WEEK #3

## 7. PUMPING LEMMA




$w = \underbrace{xyz}_{\geq N = |Q|} \in L$

CAN REPEAT y-PART
0 OR MORE TIMES

CONSEQUENCE

$$xy^i z \in L, \forall i \geq 0$$

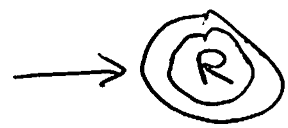
P.L. IS BASED ON PIGEONHOLE PRINCIPLE

IF HAVE N PLACES AND VISIT
THESE > N TIMES THEN MUST
REUSE AT LEAST ONE

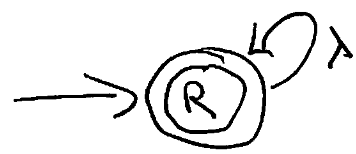
STRING OF LENGTH N MAKES N+1 VISITS
$\lambda, a_1, a_2, \ldots, a_N$

# WEEK #3

WHAT COULD HAPPEN IF ALLOWED $\lambda$-TRANSITIONS IN REG. EQ. APPROACH

CONSIDER  OVER $\Sigma = \{a\}$

$R = \lambda$ IS ONLY EQUATION AND ALL IS WELL

NOW CONSIDER  OVER $\Sigma = \{a\}$

$$R = \lambda + R \cdot \lambda$$

CLEARLY $R = \lambda$ IS $\underline{\underline{A}}$ SOLUTION

$$R = \lambda + \lambda \cdot \lambda = \lambda$$

BUT SO IS $R = \lambda + A$, WHERE $A$ IS ANY LANGUAGE OVER $\{a\}$

$$R = \lambda + (\lambda + A) \cdot \lambda = \lambda + A$$

BUT THEN THERE ARE AN UNCOUNTABLY INFINITE NUMBER OF SOLUTIONS!