Generally useful information.

- The notation **z = <x,y>** denotes the pairing function with inverses $\mathbf{x} = \mathbf{<z>_1}$ and $\mathbf{y} = \mathbf{<z>_2}$.
- The minimization notation **μ y [P(…,y)]** means the least **y** (starting at **0**) such that **P(…,y)** is true. The bounded minimization (acceptable in primitive recursive functions) notation **μ y (u≤y≤v) [P(…,y)]** means the least **y** (starting at **u** and ending at **v**) such that **P(…,y)** is true. Unlike the text, I find it convenient to define **μ y (u≤y≤v) [P(…,y)]** to be **v+1**, when no **y** satisfies this bounded minimization.
- The tilde symbol, **~**, means the complement. Thus, set **~S** is the set complement of set **S**, and predicate **~P(x)** is the logical complement of predicate **P(x).**
- The minus symbol, –, when applied to sets is set difference, so **S – T = {x | x∈S && x∉T}.**
- The absolute value, **|z|**, is the magnitude of **z**. Thus, **|x-y|** is the difference between **x** and **y**, when **x** and **y** are both non-negative.
- A function **P** is a predicate if it is a logical function that returns either **1** (**true**) or **0** (**false**). Thus, **P(x)** means **P** evaluates to **true** on **x**, but we can also take advantage of the fact that **true** is **1** and **false** is **0** in formulas like **y × P(x)**, which would evaluate to either **y** (if **P(x)**) or **0** (if **~P(x)**).
- A set **S** is recursive if **S** has a total recursive characteristic function χ_S, such that **x ∈ S** ⇔ $\chi_S(x)$. Note χ_S is a predicate. Thus, it evaluates to **0** (**false**), if **x ∉ S**.
- When I say a set **S** is re, unless I explicitly say otherwise, you may assume any of the following equivalent characterizations:
  1. **S** is either empty or the range of a total recursive function f_S.
  2. **S** is the domain of a partial recursive function g_S.
  3. **S** is recognizable by a Turing Machine.
- If I say a function **g** is partially computable, then there is an index **g** (I know that's overloading, but that's okay as long as we understand each other), such that $\Phi_g(x) = \Phi(g, x) = g(x)$. Here **Φ** is a universal partially recursive function.
  Moreover, there is a total recursive function **STP**, such that
  STP(g. x, t) is **1** (true), just in case **g**, started on **x**, halts in **t** or fewer steps.
  STP(g. x, t) is **0** (false), otherwise.
  Finally, there is another total recursive function **VALUE**, such that
  VALUE(g. x, t) is **g(x)**, whenever **STP(g. x, t)**.
  VALUE(g. x, t) is defined but meaningless if **~STP(g. x, t)**.
- The notation **f(x)↓** means that **f** converges when computing with input **x**, but we don't care about the value produced. In effect, this just means that **x** is in the domain of **f**.
- The notation **f(x)↑** means **f** diverges when computing with input **x**. In effect, this just means that **x** is **not** in the domain of **f**.
- The **Halting Problem** for any effective computational system is the problem to determine of an arbitrary effective procedure **f** and input **x**, whether or not **f(x)↓**. The set of all such pairs is a classic re non-recursive one. The set of all such **<f,x>** is denoted K_0. A related set **K** is the set of all **f** that halt on their own indices. Thus, $K = \{ f \mid \Phi_f(f)\downarrow \}$ and $K_0 = \{<f,x> \mid \Phi_f(x)\downarrow \}$
- The **Uniform Halting Problem** is the problem to determine of an arbitrary effective procedure **f**, whether or not **f** is an algorithm (halts on all input). The set of all such function indices is a classic non re one and is often called **TOTAL**.

COT 4210 Fall 2013 Final Exam Sample Questions

1. Let set **A** be recursive, **B** be re non-recursive and **C** be non-re. Choosing from among **(REC) recursive**, **(RE) re non-recursive**, **(NR) non-re**, categorize the set **D** in each of a) through d) by listing **all** possible categories. No justification is required.

a.) D = ~C ______________________________

b.) D ⊆ (A∪ C) ______________________________

c.) D = ~B ______________________________

d.) D = B – A ______________________________

2. Prove that the **Halting Problem** (the set **K_0**) is not recursive (decidable) within any formal model of computation. (Hint: A diagonalization proof is required here.)

3. Using reduction from the known undecidable **HasZero, HZ = { f | ∃x f(x) = 0 }**, show the non-recursiveness (undecidability) of the problem to decide if an arbitrary primitive recursive function **g** has the property **IsZero, Z = { f | ∀x f(x) = 0 }**.

4. Choosing from among **(D) decidable**, **(U) undecidable**, **(?) unknown**, categorize each of the following decision problems. No proofs are required.

Problem / Language Class	**Regular**	**Context Free**
L = Σ* ?		
L = φ ?		
x ∈ L^2, for arbitrary x ?		

5. Choosing from among **(Y) yes**, **(N) No**, **(?) unknown**, categorize each of the following closure properties. No proofs are required.

Problem / Language Class	**Regular**	**Context Free**
Closed under intersection?		
Closed under quotient?		
Closed under quotient with Regular languages?		
Closed under complement?		

6. Prove that any class of languages, ***C***, closed under union, concatenation, intersection with regular languages, homomorphism and substitution (e.g., the Context-Free Languages) is closed under **MissingMiddle**, where, assuming L is over the alphabet **Σ**,
MissingMiddle(L) = { xz | ∃y ∈ Σ* such that xyz ∈ L }
You must be very explicit, describing what is produced by each transformation you apply.

7. Use **PCP** to show the undecidability of the problem to determine if the intersection of two context free languages is non-empty. That is, show how to create two grammars G_A and G_B based on some instance **P** = $<<x_1,x_2,\ldots,x_n>, <y_1,y_2,\ldots,y_n>>$ of **PCP**, such that $L(G_A) \cap L(G_B) \neq \phi$ iff **P** has a solution. Assume that **P** is over the alphabet **Σ**.You should discuss what languages your grammars produce and why this is relevant, but no formal proof is required.

8. Consider the set of indices **CONSTANT = { f | ∃K ∀y [** $\varphi_f(y) = K$ **] }**. Use Rice's Theorem to show that **CONSTANT** is not recursive. Hint: There are two properties that must be demonstrated.

9. Show that **CONSTANT** $\equiv_m$ **TOT**, where **TOT = { f | ∀y** $\varphi_f(y)\downarrow$ **}**.

10. Why does Rice's Theorem have nothing to say about each of the following? Explain by showing some condition of Rice's Theorem that is not met by the stated property.
a.) **AT-LEAST-LINEAR = { f | ∀y** $\varphi_f(y)$ **converges in no fewer than y steps }**.

b.) **HAS-IMPOSTER = { f | ∃ g [g≠f and ∀y [** $\varphi_g(y) = \varphi_f(y)$ **]] }**.

11. We described the proof that **3SAT** is polynomial reducible to Subset-Sum.
a.) Describe **Subset-Sum**
b.) Show that **Subset-Sum** is in **NP**
c.) Assuming a **3SAT** expression **(a + ~b + c) (b + b + ~c)**, fill in the upper right part of the reduction from **3SAT** to **Subset-Sum**.

	a	**b**	**c**	**a + ~b + c**	**b + b + ~c**
a	**1**				
~a	**1**				
b		**1**			
~b		**1**			
c			**1**		
~c			**1**		
C1				**1**	
C1'				**1**	
C2					**1**
C2'					**1**
	1	**1**	**1**	**3**	**3**

12. Use the appropriate Pumping Lemmas to show:
a.) **{ ww | w is over {a,b} }** is not Regular
b.) **{ ww | w is over {a,b} }** is not Context Free

13. Write a context-free grammar for the complement of the language **{ ww | w is in {a,b}* }**