

Assignment # 7.1a Sample Key

1. For the following languages, either provide a grammar to show it is a CFL or employ the Pumping Lemma to show it is not

a.) $L = \{ a^i b^j \mid j > 2 \cdot i \}$

This language is a CFL. A grammar that works is

$$S \rightarrow aSbb \mid Sb \mid b$$

Assignment # 7.1b Sample Key

1. b.) $L = \{ a^n b^{\text{Fib}(n)} \mid n > 0 \}$, where **Fib(i)** is the i^{th} Fibonacci number

PL: Provides $N > 0$

We: Choose $a^N b^{N!} \in L$

PL: Splits $a^N b^{N!}$ into $uvwxy$, $|vwx| \leq N$, $|vx| > 0$, such that $\forall i \geq 0 uv^i wx^i y \in L$

We: Choose $i=2$

Case 1: vwx contains only b 's, then we are increasing the number of b 's while leaving the number of a 's unchanged. In this case $uv^2 wx^2 y$ is of form $a^N b^{N!+c}$, $c > 0$ and this is not in L .

*Case 2: vwx contains some a 's and maybe some b 's. Under this circumstances $uv^2 wx^2 y$ has at least $N+1$ a 's and at most $N!+N-1$ b 's. But $(N+1)! = N!(N+1) = N! * N + N \geq N! + N > N! + N - 1$ and so is not in L .*

Cases 1 and 2 cover all possible situations, so L is not a CFL

Assignment # 7.2 Sample Key

2. Consider the context-free grammar $G = (\{ S \}, \{ a, b \}, S, P)$, where P is:

$$S \rightarrow S a S b S \mid S b S a S \mid S a S a S \mid a \mid \lambda$$

Provide the first part of the proof that

$$L(G) = L = \{ w \mid w \text{ has at least as many } a\text{'s as } b\text{'s} \}$$

That is, show that $L(G) \subseteq L$

To attack this problem we can first introduce the notation that, for a syntactic form α , $\alpha_a =$ the number of a 's in α , and $\alpha_b =$ the number of b 's in α . Using this, we show that if $S \Rightarrow^* \alpha$, then $\alpha_b \leq \alpha_a$ and hence that $L(G) \subseteq L$:

A straightforward approach is to show, inductively on the number of steps, i , in a derivation, that, if $S \Rightarrow^i \alpha$, then $\alpha_b \leq \alpha_a$.

Assignment # 7.2 Sample Key

Basis (i=1): Since $S \Rightarrow \alpha$ iff $S \rightarrow \alpha$ and all rhs of S have $\alpha_b \leq \alpha_a$ then the base case holds

IH: Assume if $S \Rightarrow_m \alpha$, then $\alpha_b \leq \alpha_a$, whenever $m \leq n$

IS: Show that if $S \Rightarrow_{n+1} \alpha$, then $\alpha_b \leq \alpha_a$

If $S \alpha$ then $S \Rightarrow_n \beta$ and $\beta \Rightarrow \alpha$

Since G has only one non-terminal S , the rewriting of β to α involves a single application of one of the S -rules. By the I.H., β has the property that $\beta_b \leq \beta_a$. Since a single application of an S rule either adds no b 's or a 's, one a , one a and one b , or two b 's, we have the three following cases:

Assignment # 7.2 Sample key

- Case 0: $\alpha_a = \beta_a$, and $\alpha_b = \beta_b$
In which case, using the IH, we have:
 $\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$
- Case 1: $\alpha_b = \beta_b$, and $\alpha_a = \beta_a + 1$
In which case, using the IH, we have:
 $\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$
- Case 2: $\alpha_b = \beta_b + 1$, and $\alpha_a = \beta_a + 1$
In which case, using the IH, we have:
 $\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$
- Case 3: $\alpha_b = \beta_b$, and $\alpha_a = \beta_a + 2$
In which case, using the IH, we have:
 $\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$