1. **Quotient of Regular**

\[ L/R = \{ x \mid \exists y \in R, xy \in L \} \]

\[ Q_L = (Q_L, \Sigma, \delta_L, q_L, F_L) \quad L = \mathcal{L}(Q_L) \]

\[ Q_R = (Q_R, \Sigma, \delta_R, q_R, F_R) \quad R = \mathcal{L}(Q_R) \]

\[ Q_{L/R} = (Q_{L \cup Q_L \times Q_R}, \Sigma, \delta_{L/R}, q_{L/R}, F_{L/R}) \]

\[ F_{L/R} = F_L \times F_R \]

\[ \delta_{L/R}(q, a) = \{ \delta_L(q, a) \} \quad q \in Q_L, a \in \Sigma \]

\[ \delta_{L/R}(q, \lambda) = \{ <q, q_{IR}> \} \quad q \in Q_L \]

\[ \delta_{L/R}(<q, p>, a) = \{ <\delta_L(q, a), \delta_R(p, a)> \} \]

\[ q \in Q_L, p \in Q_R, a \in \Sigma \]

\[ L/R = \mathcal{L}(Q_{L/R}) \]

**WHY???
Closure Metatechnique

Assume closure under
a) Substitution / Homomorphism
b) Intersection with Regular Languages

Define
\( f(a) = \{a,a'\} \quad ; \quad g(a) = a' \)
\( h(a) = a \quad ; \quad h(a') = \lambda \)

General form of closure
\[ \text{OP}(L) = h(f(L) \cap R) \]

Regular pattern to use as filter

Example 1
Prefix(L) = \( h(f(L) \cap \Sigma^* g(\epsilon^*)) \)
\[ = h(\{xy' | xy \notin L\} \) \]
\[ = \{ x | \exists y \in \Sigma^*, xy \notin L \} \]

Example 2
\[ L/R = h(f(L) \cap \Sigma^* g(R)) \]
\[ = h(\{xy' | xy \in L, y \in R \}) \]
\[ = \{ x | \exists y \in R, xy \in L \} \]
3. More Examples:

\[ \text{SUFFIX}(L) = \{ f(L) \cap g(\Sigma^*) \Sigma^* \} \]

\[ \text{SUBSTRING}(L) = \{ f(L) \cap g(\Sigma^*) \Sigma^* g(\Sigma^*) \} \]

4. Other ways to get SUFFIX, SUBSTRING assuming PREFIX and REVERSE

\[ \text{SUFFIX}(L) = (\text{PREFIX}(L^R))^R \]

\[ \text{SUBSTRING}(L) = \text{SUFFIX}(\text{PREFIX}(L)) = \text{PREFIX}(\text{SUFFIX}(L)) \]
5. PROOF OF PREFIX(L) USING DFAs
\[ L = \mathcal{L}(\delta) \quad Q = (Q, \Sigma, \delta, q_0, F) \]
\[ \text{PREFIX}(L) = \mathcal{L}(\delta') \quad \text{WHERE} \]
\[ Q' = (Q, \Sigma, \delta', q_0, F') \]
\[ F' = \{ q \mid \exists w \quad \delta'(q, w) \notin F, w \in \Sigma^* \} \]

5. PROOF OF SUFFIX(L) USING DFAs
\[ L = \mathcal{L}(\delta) \quad Q = (Q, \Sigma, \delta, q_0, F) \]
\[ \text{SUFFIX}(L) = \mathcal{L}(\delta'') \quad \text{WHERE} \]
\[ Q'' = (Q, \Sigma, \delta'', F) \]
\[ S'' = \{ q \mid \exists w \quad \delta''(q, w) = q, w \in \Sigma^* \} \]

REALLY WE ARE LIMITED TO ONE START STATE IN FORMAL MODEL, SO CREATE NEW START STATE SO
\[ Q'' = (Q \cup \{s_0\}, \Sigma, \delta'', s_0, F) \]
\[ S''(q, a) = \{ w \mid \delta''(q, a) \cdot w \in \Sigma^* \} \]
\[ S''(s_0, \lambda) = \{ w \mid \delta''(s_0, \lambda) \cdot w \in \Sigma^* \} \]

THIS IS AN NFA
7. **Pumping Lemma for Regular Languages**

Let L be regular then

\[ \exists n > 0 \text{ such that, if } w \in L \text{ and } |w| \geq n \text{ then } w \text{ can be written } \]

in form \( x_0 y_1 z \) \( (w = x_0 y_1 z) \)

where \( |x_0 y_1| \leq n, |y_1| \geq 0 \) and \( \forall i \geq 0 \ x_0 y_1^i z \in L \)

Uses "Pigeon Hole Principle."

If I have n containers

and I have \( m > n \) items to put in containers then at least one container must accommodate more than one item.
8. Pumping Lemma in Detail

Let \( L = \mathcal{L}(Q) \) where \( Q = (Q, \Sigma, S, q_0, F) \)

Let \( N = \lVert Q \rVert \)

Let \( w \in \Sigma^+ \) where \( \lVert w \rVert \geq N \)

\( w = v_1 v_2 \cdots v_m \) \( m > N \) \( \forall i \in \Sigma \)

As \( Q \) starts in \( q_0 \), it visits one state prior to reading \( v_1 \)

At or before reading \( v_N \), \( Q \) must visit at least one state more than once

Let \( v_1 \cdots v_j \) be shortest string to visit some repeated state, \( q_r \)

And \( v_{j+1} \cdots v_k \) be shortest non-empty string to revisit \( q_r \)

So \( S^*(q_0, v_1 \cdots v_j) = q_r \)

\( S^*(q_0, v_{j+1} \cdots v_k) = q_r \)

\( S^*(q_r, v_{k+1} \cdots v_m) \in F \)

Let \( x = v_1 \cdots v_j \) \( y = v_{j+1} \cdots v_k \)

\( z = v_{k+1} \cdots v_m \)

Then \( S^*(q_0, x y_i^i) = q_r \) \( \forall i > 0 \)

And \( S^*(q_0, x y_i^i z) \in F \) \( \forall i > 0 \)
9. Pumping Lemma Application

Using Adversarial Process

Me: Assume $L = \{a^nb^n | n \geq 0\}$ Regular

P & L: Provides $n > 0$. I can assume nothing about $n$ except $n > 0$

Me: Choose some string with length $\geq n$. I'll choose $a^nb^n$

P & L: $a^nb^n = xy^i z$, $|xy| \leq n, |y| \geq 0$

And $\forall i \geq 0$, $xy^i z \in L$

I cannot choose split but P & L is constrained by $n$

Me: I choose $i = 0$

I can choose any $i > 0$

P & L: From above must assert $a^{n-1}y_1 b^n \in L$

Me: Gotcha as with $|y| > 0$

$n - |y| \neq n$ so $a^{n-1}y_1 b^n \notin L$

Contradiction so $L$ is not regular
The Myhill-Nerode Theorem

Based on notion of right-invariant equivalence relations over strings

\[ x R y \Rightarrow \forall z \; x R y \Rightarrow x y z \in \Sigma^* \]

Consider \( Q = (Q, \Sigma, S, q_0, F) \), a DFA.

Define \( R_Q \) by

\[ x R_Q y \iff S^*(q_0, x) = S^*(q_0, y) \]

\( R_Q \) is an equivalence relation and it is right-invariant, as

\[ S^*(q_0, x) = S^*(q_0, y) \]

\[ \Rightarrow \forall z \; S^*(q_0, x z) = S^*(q_0, y z) \]

\[ \Rightarrow \forall z \; x z R_Q y z \]

Moreover, the index of \( R_Q \)

(\# of partitions induced by \( R_Q \))

is finite (1\(|Q|\))
11. MY HILL-WERODE THEOREM STATEMENT

(a), (b), & (c) are equivalent

(a) \( L \) is regular

(b) \( L \) is the union of some of the equivalence classes of some R.I.E.R. of finite index

(c) The specific R.I.E.R. \( R_L \) defined by

\[ xR_Ly \iff \forall \beta [\exists \beta L \iff y \beta L] \]

has finite index
13. MYHILL–NEERO DE PROOF (LAST STAGE)

(c) \Rightarrow (a)

Define \( Q_{RL} = (Q, \Sigma, S, \{x\}, F) \)

\( Q = \{ [x]_{RL} | x \in \Sigma^* \} \) finite

\( F = \{ [x]_{RL} | x \in L \} \) subset of \( Q \)

Note: \([x]_{RL}\) is equiv. class containing \( x\)

\( \delta([x], a) = [x \cdot a] \)

14. CONSEQUENCES

A. MIN DFA IS UNIQUE AS ALL DFAS FOR \( L \) ARE REFINEMENTS OF ONE DERIVED FROM \( R_L \)

B. A LANGUAGE \( L \) IS NOT REGULAR IF \( R_L \) HAS INFINITE INDEX
L = \{a^n b^n | n \geq 0\} is NOT REG.

Consider \([a^i]^n_{RL} \mid i > 0\)

\(a^i b^i \in L\) but
\(a^i b^j \notin L\) when \(j \neq i\)

Thus, \([a^i]^n_{RL}\) and \([a^j]^n_{RL}\) are distinct when \(i \neq j\)

So \(RL\) has infinite index

\(L = \{a^{n^2} | n \geq 0\}\) is NOT REG.

Consider \([a^{i^2}]^n_{RL} \mid i > 0\)

\(a^{i^2} a^{2i+1} \in L\) but
\(a^{j^2} a^{2j+1} \notin L\) when \(j > i\)

Thus, \([a^{i^2}]^n_{RL}\) and \([a^{j^2}]^n_{RL}\) are distinct when \(i \neq j\)

(i > j or i < j)

So \(RL\) has infinite index