Assignment # 7.1a Sample Key

1. For the following languages, either provide a grammar to show it is a CFL or employ the Pumping Lemma to show it is not

a.) $L = \{ a^i b^j \mid j > 2i \}$

This language is a CFL. A grammar that works is

$S \rightarrow aSbb \mid Sb \mid b$
1. b.) \( L = \{ a^n b^n! \mid n>0 \} \)

**PL:** Provides \( N>0 \)

**We:** Choose \( a^N b^N! \in L \)

**PL:** Splits \( a^N b^N! \) into \( uvwx \), \( |vwx| \leq N \), \( |vx| > 0 \), such that \( \forall i \geq 0 \ uv^iwx^iy \in L \)

**We:** Choose \( i=2 \)

**Case 1:** \( vx \) contains only \( b \)'s, then we are increasing the number of \( b \)'s while leaving the number of \( a \)'s unchanged. In this case \( uv^2wx^2y \) is of form \( a^N b^{N+c} \), \( c>0 \) and this is not in \( L \).

**Case 2:** \( vx \) contains some \( a \)'s and maybe some \( b \)'s. Under this circumstances \( uv^2wx^2y \) has at least \( N+1 \) \( a \)'s and at most \( N!+N-1 \) \( b \)'s. But \( (N+1)! = N!(N+1) = N!*N+N \geq N! + N > N!+N-1 \) and so is not in \( L \).

**Cases 1 and 2** cover all possible situations, so \( L \) is not a CFL
2. Consider the context-free grammar $G = (\{S\}, \{a, b\}, S, P)$, where $P$ is:

$$S \rightarrow S \ a \ S \ b \ S \ | \ S \ b \ S \ a \ S \ | \ S \ a \ S \ a \ S \ | \ a \ | \lambda$$

Provide the first part of the proof that $L(G) = L = \{w | w \text{ has at least as many } a's \text{ as } b's \}$

That is, show that $L(G) \subseteq L$

To attack this problem we can first introduce the notation that, for a syntactic form $\alpha$, $\alpha_a = \text{the number of } a's \text{ in } \alpha$, and $\alpha_b = \text{the number of } b's \text{ in } \alpha$. Using this, we show that if $S \Rightarrow^* \alpha$, then $\alpha_b \leq \alpha_a$ and hence that $L(G) \subseteq L$:

A straightforward approach is to show, inductively on the number of steps, $i$, in a derivation, that, if $S \Rightarrow_i \alpha$, then $\alpha_b \leq \alpha_a$. 
Assignment # 7.2 Sample Key

Basis (i=1): Since $S \Rightarrow \alpha$ iff $S \rightarrow \alpha$ and all rhs of $S$ have $\alpha_b \leq \alpha_a$ then the base case holds

IH: Assume if $S \Rightarrow_m \alpha$, then $\alpha_b \leq \alpha_a$, whenever $m \leq n$

IS: Show that if $S \Rightarrow_{n+1} \alpha$, then $\alpha_b \leq \alpha_a$

If $S \alpha$ then $S \Rightarrow_n \beta$ and $\beta \Rightarrow \alpha$

Since G has only one non-terminal $S$, the rewriting of $\beta$ to $\alpha$ involves a single application of one of the $S$-rules. By the I.H., $\beta$ has the property that $\beta_b \leq \beta_a$. Since a single application of an $S$ rule either adds no $b$’s or $a$’s, one $a$, one $a$ and one $b$, or two $b$’s, we have the three following cases:
Assignment # 7.2 Sample key

Case 0: \( \alpha_a = \beta_a \), and \( \alpha_b = \beta_b \)
In which case, using the IH, we have:
\( \beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a \)

Case 1: \( \alpha_b = \beta_b \), and \( \alpha_a = \beta_a + 1 \)
In which case, using the IH, we have:
\( \beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a \)

Case 2: \( \alpha_b = \beta_b + 1 \), and \( \alpha_a = \beta_a + 1 \)
In which case, using the IH, we have:
\( \beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a \)

Case 3: \( \alpha_b = \beta_b \), and \( \alpha_a = \beta_a + 2 \)
In which case, using the IH, we have:
\( \beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a \)