M-N

ALL ARE EQUIV.

1. L IS $\mathcal{L}(a)$ FOR SOME DFA

$a = (Q, \Sigma, \delta, q_0, F)$

ASSUME ALL STATES IN Q ARE REACHABLE

2. L IS THE UNION OF SOME OF THE CLASSES OF A R.I.E.R R THAT IS OF FINITE INDEX

R_a: $x R_a y \Leftrightarrow \delta^*(q_0, x) = \delta^*(q_0, y)$

R.I.: $x R_a y \Rightarrow \forall z \; xz R_a yz$

L = THE UNION OF ALL $[x]$, $x \in \mathcal{L}(a)$

$$L = \bigcup_{q \in F} \{ [x] \mid \delta(q_0, x) = q \}$$

FOR ODD PARITY [0] [1]

3. THE SPECIFIC R.I.E.R

R_L: $x R_L y \Leftrightarrow \forall z \, [xz \in L \Leftrightarrow yz \in L]$

R_a: $x R_a y \Rightarrow \forall z \, [xz \in L \Leftrightarrow yz \in L]$

MYHILL-NERODE AND WHAT IS NOT REGULAR

FOR ANY LANGUAGE L, DEFINE R_L FROM MYHILL-NERODE, THAT IS,

$$xR_Ly \Leftrightarrow \forall z\ [xz \in L \Leftrightarrow yz \in L]$$

IF R_L HAS FINITE INDEX, L IS REGULAR, ELSE L IS NOT REGULAR

E.G., LET $L = \{x \mid x \in \{0,1\}^*$ AND X IS ODD PARITY$\}$

TWO CLASSES INDUCED BY R_L ARE

$[\lambda] = \{x \mid x \in \{0,1\}^*$ AND X HAS EVEN PARITY$\}$

$[1] = \{x \mid x \in \{0,1\}^*$ AND X HAS ODD PARITY$\}$

$L = [1]$

INDEX IS 2

SO MINIMAL STATE DFA HAS TWO STATES

SECOND POSITIVE EXAMPLE FROM M-N

DEFINE

$L = \{x \mid x \in \{0,1\}^+ \text{ AND } x \bmod 5 = 1 \text{ OR } x \bmod 5 = 3\}$

CLASSES (PARTITIONS ARE)

$[\lambda] = \{\lambda\}$

$[0] = \{x \mid x \in \{0,1\}^+ \text{ AND } x \bmod 5 = 0\}$

$[1] = \{x \mid$ " " $x \bmod 5 = 1\}$

$[2] = \{x \mid$ " " $x \bmod 5 = 2\}$

$[3] = \{x \mid$ " " $x \bmod 5 = 3\}$

$[4] = \{x \mid$ " " $x \bmod 5 = 4\}$

$L = [1] \cup [3]$

AND R_L HAS INDEX 6 WHICH IS FINITE

NOTE: THIS IS HOW I TEND TO NAME STATES OF DFAs

EXAMPLE WHERE M-N SHOWS L NON-REG,

$L = \{a^n b^n \mid n > 0\}$

NOTE! DO <u>NOT</u> NEED TO SHOW ALL CLASSES. IT IS SUFFICIENT TO SHOW AN INFINITE NUMBER OF PARTITIONS OF SOME EASILY SHOWN FORM

CONSIDER $[a^k b]_{R_L}$ $k > 0$

FOR EACH $i, j > 0$ AND $i \neq j$

CONSIDER $[a^i b]_{R_L}$ AND $[a^j b]_{R_L}$

$a^i b \cdot b^{i-1} \in L$

BUT $a^j b \cdot b^{i-1} = a^j b^i, i \neq j, \notin L$

SO, FOR EACH DISTINCT $k > 0$, $[a^k b]_{R_L}$ IDENTIFIES A UNIQUE PARTITION, AND THUS THERE ARE AN INFINITE NUMER OF SUCH PARTITIONS AND SO R_L HAS INFINITE INDEX AND L IS NOT REGULAR.

M-N AGAIN

$L = \{xax \mid x \in \{a,b\}^+\}$

CONSIDER $[a^i b]_{R_L}$ $i > 0$

$a^i baa^i b \in L$, BUT

$a^k baa^i b \notin L$, $k < i$

AND SO $[a^i b]_{R_L} \neq [a^k b]_{R_L}$ $k < i$

THUS, $[a^i b]_{R_L}$ IS DISTINCT FOR EACH $i > 0$,

AS EACH $[a^i b]_{R_L}$ IS DISTINCT, $i > 0$,

R_L'S INDEX IS INFINITE AND

L IS NOT REGULAR

M-N YET AGAIN

$L = \{a^{FIB(k)} \mid k > 0\}$

CONSIDER $[a^{FIB(j)}]_{R_L}$, $j > 2$

IT'S CLEAR
$a^{FIB(j)} a^{FIB(j+1)} = a^{FIB(j)+FIB(j+1)}$

IS IN L

BUT $a^{FIB(k)} a^{FIB(j+1)} \notin L$ WHEN
$k > 2$, $k \neq j$, AND $k \neq j+2$

SO $[a^{FIB(j)}]_{R_L}$ IS UNIQUE FOR EACH $j > 2$

THIS MEANS INF. INDEX FOR R_L

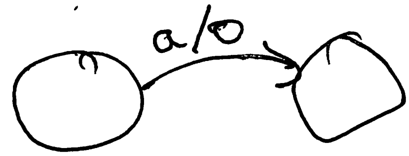
AND SO L IS NOT REGULAR

FINITE STATE MACHINES

TRANSDUCERS

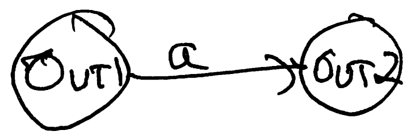
TWO MODELS

MEALY



SYNCHRONOUS

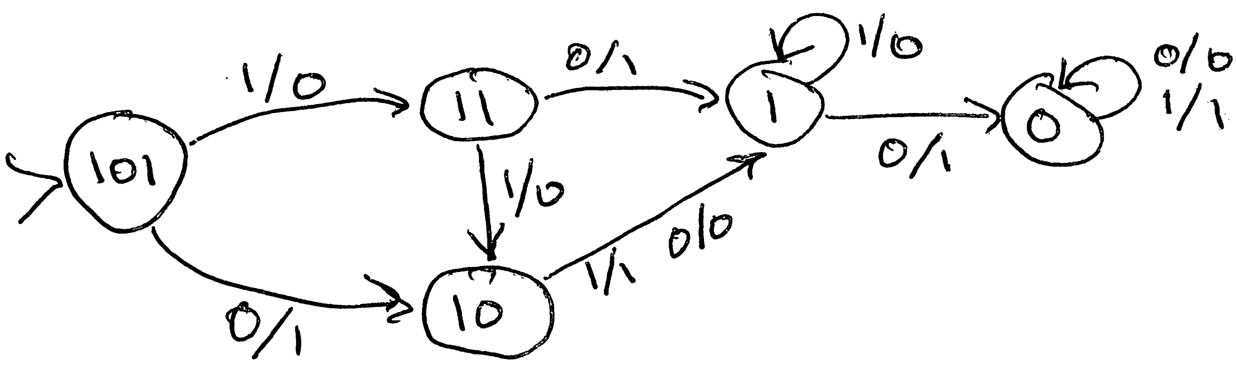
MOORE



ASYNCHRONOUS

WE FOCUS ON MEALY (OUTPUT ON TRANSITIONS ONLY)

EXAMPLE: ADD 101 TO INPUT, WHERE INPUT IS READ LSB TO MSB



TERM REWRITING SYSTEMS

DESCRIBED BY PAIR (Σ, R)

Σ IS FINITE ALPHABET OVER WHICH TERMS ARE FORMED

R IS SET OF RULES OF FORM

$$\alpha \to \beta$$

DERIVATION START WITH SOME INITIAL EXPRESSION, SAY e_0 ~~WE THEN HAVE~~ NOTION OF REWRITING, WHERE IF SOME EXPRESSION e_i HAS BEEN DERIVED THEN

$$e_i \Rightarrow e_{i+1} \quad \text{IF } \alpha \to \beta \in R$$

AND $e_i = \delta\alpha\gamma$, $e_{i+1} = \delta\beta\gamma$

$\overset{*}{\Rightarrow}$ IS THEN REFLEXIVE TRANSITIVE CLOSURE OF $\Rightarrow$

EXAMPLE TERM REWRITING

R: $x+0 \Leftrightarrow x$

$x*1 \Leftrightarrow x$

$x+y \Leftrightarrow y+x$

$\Big\}$ TWO-WAY RULES

START WITH $0+a*1$

$0+a*1 \Rightarrow 0+a$

$\Rightarrow a+0$

$\Rightarrow a$

NOTE: INHERENT IN THIS IS UNDERSTANDING OF "TERMS"

PURE REWRITING MUST USE LOTS OF CONTEXT IN RULES

REWRITING SYSTEMS CAME FROM MATHEMATICIANS TRYING TO FORMALIZE ALGORITHMIC AND LOGICAL REASONING. BIG NAMES ARE THUE AND POST

LINGUISTS CO-OPTED REWRITING SYSTEMS TO DESCRIBE LANGUAGE STRUCTURES. TO DO SO, THEY INTRODUCED A DISTINCTION BETWEEN VARIABLES AND TERMINALS

VARIABLES (AKA NON-TERMINALS) DESCRIBE SYNTACTIC CLASSES (NOUN PHRASES, NOUNS, ETC.)

TERMINALS ARE THE ELEMENTS OF ACTUAL SENTENCES. BIG NAME IS CHOMSKY

COMPUTER SCIENTISTS CO-OPTED GRAMMARS TO DESCRIBE PROGRAMMING LANGUAGES, SEQUENTIAL CIRCUIT LANGUAGES AND EVEN DATA CHECKING AND TEXTUAL ANALYSIS

GRAMMARS

$S \in V$ IS START SYMBOL

$$G = (V, \Sigma, R, S)$$

FINITE SET OF VARIABLES

FINITE SET OF TERMINALS (THE ALPHABET)

FINITE SET OF REWRITING RULES

R CONSISTS OF RULES OF FORM

$$\alpha \to \beta \qquad \alpha \in (V \cup \Sigma)^* V (V \cup \Sigma)^*$$
$$\beta \in (V \cup \Sigma)^*$$

WE START SEQUENCE OF PRODUCTION OF WORDS OVER $\Sigma \cup V$ WITH S

EACH STEP INVOLVES REWRITING PREVIOUS WORD OF FORM

$$\delta \alpha \gamma \Rightarrow \delta \beta \gamma \quad \text{WHERE } \alpha \to \beta \in R$$

PROCESS STOPS IF NO RULE APPLIES

NOTE: IF WORD OVER Σ THEN NO RULE CAN APPLY

DERIVATIONS

A DERIVATION IS A SEQUENCE OF REWRITINGS. WE DENOTE THIS AS $\overset{*}{\Rightarrow}$ THE REFLEXIVE, TRANSITIVE CLOSURE OF $\Rightarrow$

A DERIVATION LOOKS LIKE

$$w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \cdots \Rightarrow w_k \Rightarrow w_{k+1}$$

WE CAN USE NOTATION $\overset{k}{\Rightarrow}$ TO INDICATE NUMBER OF STEPS

NOTE:

$$w_1 \overset{0}{\Rightarrow} w_1$$

IF $w_1 \overset{j}{\Rightarrow} w_{j-1}$

THEN, IF $w_{j-1} = \delta \alpha \gamma$

AND $\alpha \to \beta \in R$

$$w_1 \overset{j}{\Rightarrow} \delta \beta \gamma$$

AND SO

$$w_1 \overset{*}{\Rightarrow} \delta \beta \gamma = w_j \quad j \geq 1$$

LANGUAGE ASSOC WITH GRAMMAR

IF $G = (V, \Sigma, R, S)$

THEN $\mathcal{L}(G) = \{ w \mid w \in \Sigma^* \,\&\, S \stackrel{*}{\Rightarrow} w \}$

WE CALL MEMBERS OF Σ^*
TERMINAL STRINGS OR SENTENCES

WE CALL MEMBERS OF $(V \cup \Sigma)^* V (V \cup \Sigma)^*$
SENTENTIAL FORMS

NOTE: IN COMPILERS, WE CALL
ELEMENTS OF Σ TOKENS OR
LEXEMES OR LEXICAL ITEMS
WE CALL ELEMENTS OF V
SYNTACTIC ITEMS OR VARIABLES

TYPICAL SENTENTIAL FORM MIGHT
BE

IF (<EXPRESSION>)<STATEMENT>

WHERE IF (AND) ARE TOKENS
AND <EXPRESSION> AND <STATEMENT>
ARE SYNTACTIC ITEMS

GRAMMAR HIERARCHY

TYPE 0: PHRASE STRUCTURED (PSG)

NO CONSTRAINTS ON RULES

TYPE 1: CONTEXT SENSITIVE (CSG)

IF $\alpha \to \beta \in R$ THEN $|\alpha| \leq |\beta|$

TYPE 2: CONTEXT FREE (CFG)

IF $\alpha \to \beta \in R$ THEN $\alpha \in V$

i.e., $|\alpha| = 1$

TYPE 3: RIGHT LINEAR (REGULAR)

ALL RULES OF FORMS

$A \to a \quad A \in V, a \in \Sigma$

$A \to aB \quad A, B \in V, a \in \Sigma$

LANGUAGE HIERARCHY

TYPE 0: PHRASE STRUCTURED OR RECURSIVELY ENUMERABLE

TYPE 1: CONTEXT SENSITIVE

TYPE 2: CONTEXT FREE

TYPE 3: REGULAR

FOR CFL & REGULAR, WE ALLOW $A \to \lambda$ AS IT ONLY ADDS ABILITY TO INCLUDE λ IN LANGUAGE

FOR CSL, WE ALLOW $S \to \lambda$ SO LONG AS S NEVER APPEARS OF RIGHT HAND SIDE (RHS) OF ANY RULES

EXAMPLE REGULAR GRAMMAR

$(0+1)^* 11 (0+1)^*$

$G = (\{S,T,U\}, \{0,1\}, R, S)$

R: $S \to 0S \mid 1S \mid 1T$

$T \to 1U$

$U \to 0U \mid 1U \mid \lambda$

CAN REMOVE λ-RULE BY

R': $S \to 0S \mid 1S \mid 1T$

$T \to 1U \mid 1$

$U \to 0U \mid 1U \mid 0 \mid 1$

ODD PARITY $G = (\{\langle EVEN \rangle, \langle ODD \rangle\}, \{0,1\}, R, \langle EVEN \rangle)$

$\langle EVEN \rangle \to 0\langle EVEN \rangle \mid 1\langle ODD \rangle$

$\langle ODD \rangle \to 0\langle ODD \rangle \mid 1\langle EVEN \rangle \mid \lambda$

REGULAR GRAMMARS AND CLOSURE

$G_1 = (V_1, \Sigma, R_1, S_1)$ $\quad V_1 \cap V_2 = \emptyset$

$G_2 = (V_2, \Sigma, R_2, S_2)$

UNION $G_3 = (V_1 \cup V_2 \cup \{S_3\}, \Sigma, R_1 \cup R_2 \cup R_3, S_3)$ $\quad S_3 \notin V_1 \cup V_2$

WHERE R_3 IS

$S_3 \to rhs(S_1)$

$\quad | \; rhs(S_2)$

rhs IS RIGHT-HAND SIDES OF A VARIABLE'S RULES

i.e., IF $S_1 \to \alpha_1 | \alpha_2 | \cdots | \alpha_k$ THEN rhs IS $\alpha_1, \alpha_2, \ldots, \alpha_k$ AND WE WOULD HAVE

$S_2 \to \alpha_1 | \alpha_2 | \cdots | \alpha_k$

CONCAT $G_4 = (V_1 \cup V_2, \Sigma, R_4 \cup R_2, S_1)$

$A \to rhs(S_2)$ WHENEVER $A \to \lambda \in R_1$

R_4: $A \to a S_2$ WHENEVER $A \to a \in R_1$, $a \in \Sigma$

$\cup \; R_1 - \{A \to a \mid A \to a \in R_1 \;\&\; a \in \Sigma \cup \{\lambda\}\}$

STAR

$G_5 = (V_1, \Sigma, R_1 \cup R_5, S_5)$

R_5: $A \to a S_1$ WHENEVER $A \to a \in R_1$

$S_5 \to \lambda \,|\, rhs(S_1)$

REGULAR GRAMMARS AND REGULAR LANGUAGES

CLEARLY REGULAR GRAMMARS GENERATE ALL REGULAR LANGUAGES SINCE CAN GET BASE LANGUAGES

$S \to a \qquad a \in \Sigma \qquad L = \{a\}$

$S \to \lambda \qquad L = \{\lambda\}$

NO S RULES $\qquad L = \emptyset$

AND GRAMMARS ARE CLOSED UNDER $\cup, \circ, *$

A CONSTRUCTIVE APPROACH IS

$\mathcal{Q} = (Q, \Sigma, \delta, q_0, F)$, $\mathcal{Q}$ IS A DFA

$G_{\mathcal{Q}} = (Q, \Sigma, R_{\mathcal{Q}}, q_0)$

SEE NEXT PAGE

CONVERT DFA TO REGULAR GRAMMAR

$\mathcal{Q} = (Q, \Sigma, \delta, q_0, F)$

$G_{\mathcal{Q}} = (Q, \Sigma, R_{\mathcal{Q}}, q_0)$

WHENEVER $\delta(p, a) = q$

$p \to a q \in R_{\mathcal{Q}}$

WHENEVER $f \in F$

$f \to \lambda \in R_{\mathcal{Q}}$

$R_{\mathcal{Q}}$ CONTAINS NO OTHER RULES

CLAIM:

$\delta^*(q_0, w) = t \quad w \in \Sigma^*, t \in Q$

IFF $q_0 \overset{*}{\underset{G_{\mathcal{Q}}}{\Rightarrow}} w t$

& $\delta^*(q_0, w) \in F$

IFF $q_0 \overset{*}{\underset{G_{\mathcal{Q}}}{\Rightarrow}} w$

CONVERT GRAMMAR TO NFA

$G = (V, \Sigma, R, S)$

$\mathcal{A}_G = (V', \Sigma, \delta_G, S, \{f\})$

$V' = V \cup \{f\}$

WHENEVER $A \to a \quad a \in \Sigma \cup \{\lambda\}$

$\delta_G(A, a) \supseteq \{f\}$

WHENEVER $A \to aB \quad B \in V, a \in \Sigma$

$\delta_G(A, a) \supseteq \{B\}$

δ_G HAS NO OTHER TRANSITIONS

CLAIM:

$S \stackrel{*}{\Rightarrow} wX, \; w \in \Sigma^*, \; X \in V$

IFF $\delta_G^*(S, w) \supseteq \{X\}$

& $S \stackrel{*}{\Rightarrow} w, \; w \in \Sigma^*$

IFF $\delta_G^*(S, w) \supseteq \{f\}$

REGULAR LANGUAGES & LEFT LINEAR GRAMMARS

LET $G = (V, \Sigma, R, S)$ BE LEFT-LINEAR

i.e. RULES IN R ARE ALL FORM

$$A \to a \qquad A \in V$$
$$A \to Ba \qquad B \in V$$
$$A \to \lambda \qquad a \in \Sigma$$

IF TAKE G AND REVERSE ALL RULES IN R, i.e., IF $A \to Ba$, REPLACE BY $A \to aB$, RESULT IS RIGHT LINEAR AND NEW GRAMMAR PRODUCES L^R WHERE $L = \mathcal{L}(G)$. BUT REGULAR CLOSED UNDER REVERSAL, SO L IS REGULAR. THUS, ALL LEFT-LINEAR GRAMMARS GENERATE REGULAR LANGUAGES AND SO ARE ALSO CLOSED UNDER REVERSAL.

THIS ALLOWS US TO SHOW EQUIVALENCE SINCE THE REVERSAL OF EVERY RIGHT LINEAR GRAMMAR IS LEFT LINEAR AND VICE-VERSA

MIXING RIGHT & LEFT LINEAR

$G = (\{S, T\}, \{a, b\}, R, S)$

$S \to aT$

$T \to Tb \mid b$

THEN $\mathcal{L}(G) = \{a^n b^n \mid n > 0\}$

AND SO IS NON-REGULAR

THUS, MIXING THEM GETS US

INTO DOMAIN OF CONTEXT FREE

REGULAR

RECOGNIZED BY

DFA

NFA

DENOTED BY

REG EXP

SOLUTION TO

REG. EQUATIONS

GENERATED BY

REG. GRAMMAR

(RIGHT LINEAR)

LEFT LINEAR

FINISHING UP

$\text{MAX}(L) = \{w \mid w \in L$ & w IS NOT THE PROPER PREFIX OF ANY WORD IN $L\}$

$= \{w \mid w \in L$ AND IF $y \in \Sigma^*$, $wy \notin L\}$

$\text{MIN}(L) = \{w \mid w \in L$ AND NO PROPER PREFIX OF w IS IN $L\}$

$= \{w \mid w \in L$ AND IF $w = xy$, $x \in \Sigma^*$, $y \in \Sigma^+$ THEN $x \notin L\}$

$\text{MIN}(0(0+1)^*) = \{0\}$

$\text{MAX}(0(0+1)^*) = \{\}$

$\text{MIN}(01+0+10) = \{0, 10\}$

$\text{MAX}(01+0+10) = \{01, 10\}$

$\text{MIN}(\{a^i b^j c^k \mid i \leq k \text{ OR } j \leq k\}) =$
$\{a^i b^j c^k, k = \min(i,j)\}$

$\text{MAX}(\{a^i b^j c^k \mid i \leq k \text{ OR } j \leq k\}) = \{\}$

$\text{MIN}(\{a^i b^j c^k \mid i \geq k \text{ OR } j \geq k\}) = \{\lambda\}$

$\text{MAX}(\{a^i b^j c^k \mid i \geq k \text{ OR } j \geq k\}) =$
$\{a^i b^j c^k \mid i, j \geq 0, k = \text{MAX}(i,j)\}$

REACHINGTO REACHINGFROM

LET $\mathcal{Q} = (Q, \Sigma, \delta, q_0, F)$ THEN

FOR EACH $q \in Q$

$\text{REACHINGTO}(q) = \{p \mid p \in Q, \exists w \in \Sigma^*, \delta^*(p, w) = q\}$

$\text{REACHINGFROM}(q) = \{p \mid p \in Q, \exists w \in \Sigma^*, \delta^*(q, w) = p\}$

BOTH CAN BE COMPUTED USING DEPTH-FIRST SEARCH

WE USE $\text{REACHINGFROM}(q_0)$ TO DETERMINE USEFUL STATES OF $\mathcal{Q}$. ALSO USEFUL FOR MAX AND MIN

$\mathcal{Q}_{MAX} = (Q, \Sigma, \delta, q_0, F')$

WHERE $F' = \{f \mid f \in F, \text{REACHINGFROM}^+(f) \cap F = \emptyset\}$

HERE $\text{REACHINGFROM}^+(q) = \{p \mid p \in Q, \exists w \in \Sigma^+, \delta^+(q, w) = p\}$

$\mathcal{Q}_{MIN} = (Q, \Sigma, \delta, q_0, F')$

WHERE $F' = \{f \mid f \in F, \text{REACHINGTO}^+(f) \cap F = \emptyset\}$

HERE $\text{REACHINGTO}^+(q) = \{p \mid p \in Q, \exists w \in \Sigma^+, \delta^+(p, w) = q\}$

$$\tfrac{1}{2}(L) = \{x \mid \exists y, |x| = |y| \,\&\, xy \in L\}$$

$$Q = (Q, \Sigma, \delta, q_0, F) \qquad L = \mathcal{L}(Q)$$

GUESS STATE AT MID	CHANNEL 3
USE GUESS TO GUESS END PART	CHANNEL 2
OPERATE AS NORMAL FOR FIRST PART	CHANNEL 1

STATES ARE FROM $Q \times Q \times Q \cup \{s\}$

S IS NEW START THAT GOES TO ALL GUESSES

$$\delta_{1/2}(s, \lambda) = \langle q_0, q, q \rangle \qquad q \in Q$$

$$F_{1/2} = \{\langle t, f, t \rangle \mid t \in Q, f \in F\}$$

TRANSITION FUNCTION ON NEXT PAGE

$A = (Q, \Sigma, \delta, q_0, F)$

$\delta_{1/2}(\langle q, r, t\rangle, a) =$

$\langle \delta(q,a), u, t \rangle$

WHERE

U IS ANY STATE GOTTEN FROM r WITH ANY INPUT FROM Σ

i.e.,

$$\delta_{1/2}(\langle q, r, t\rangle, a) = \bigcup_{b\in\Sigma} \langle \delta(q,a), \delta(r,b), t \rangle \quad \forall b \in \Sigma$$

WHERE $q, r, t \in Q$ & $a, b \in \Sigma$

$A_{1/2} = (Q \cup \{s\}, \Sigma, \delta_{1/2}, s, F_{1/2})$

SEE PREV. PAGE

REGULAR CLOSED UNDER 1/2

REVIEW

MONDAY 9/24

10-12

HEC-101

AND IN-CLASS

TUESDAY 9/25