Assignment # 8.1 Key

1. Show that Halt reduces to Non-Trivial, where
   Non-Trivial = { f | for some x and y, x ≠ y, $\varphi_f(x)\downarrow$, $\varphi_f(y)\downarrow$ and $\varphi_f(x) \neq \varphi_f(y)$ }

Let f be an arbitrary natural number. f is in Non-Trivial iff for some x and y, x ≠ y, $\varphi_f(x)\downarrow$, $\varphi_f(y)\downarrow$ and $\varphi_f(x) \neq \varphi_f(y)$

Define g by $\varphi_g(z) = \exists$<x,y,t> [STP(f,x,t) & STP(f,y,t) & (x≠y) & (VALUE(f,x,t) ≠ (VALUE(f,y,t))], for all z.

Clearly, $\varphi_g(z) = 1$, for all z and so <g,0> is in Halt, iff there is some pair, x,y, x≠y, such that $\varphi_f(x)\downarrow$ and $\varphi_f(y)\downarrow$ and $\varphi_f(x) \neq \varphi_f(y)$, and $\varphi_g(z)\uparrow$, for all z, otherwise, and so <g,0> is not in Halt.

Summarizing, f is in Non-Trivial iff g is in Halt and so

Non-Trivial $\leq_m$ Halt as we were to show.

Assignment # 8.2 Key

2. Show that Non-Trivial reduces to Halt. (1 plus 2 show they are equally hard)

Let f be an arbitrary natural number. f is in Non-Trivial iff for some x and y, $x \neq y$, $\varphi_f(x)\downarrow$, $\varphi_f(y)\downarrow$ and $\varphi_f(x) \neq \varphi_f(y)$

Define g by $\varphi_g(z) = \exists$<x,y,t> [STP(f,x,t) & STP(f,y,t) & (x≠y) & (VALUE(f,x,t) ≠ (VALUE(f,y,t))], for all z.

Clearly, $\varphi_g(z) = 1$, for all z and so <g,0> is in Halt, iff there is some pair, x,y, x≠y, such that $\varphi_f(x)\downarrow$ and $\varphi_f(y)\downarrow$ and $\varphi_f(x) \neq \varphi_f(y)$, and $\varphi_g(z)\uparrow$, for all z, otherwise, and so <g,0> is not in Halt.

Summarizing, f is in Non-Trivial iff g is in Halt and so

Non-Trivial $\leq_m$ Halt as we were to show.

Assignment # 8.3 Alternate Key

3. Use Reduction from Total to show that NoRepeats is not even re, where
   NoRepeats = { f | for all x and y, $\varphi_f(x)\downarrow$, $\varphi_f(y)\downarrow$,
   and $x \neq y \Rightarrow \varphi_f(x) \neq \varphi_f(y)$ }

Let f be an arbitrary natural number. f is in Total iff $\forall x\ \varphi_f(x)\downarrow$

Define g by $\varphi_g(x) = \varphi_f(x) - \varphi_f(x) + x$ for all x.

Clearly, $\varphi_g(x) = x$, for all x, (converges everywhere with no repetitions) iff $\forall x\ \varphi_f(x)\downarrow$; otherwise $\varphi_g(x)\uparrow$ for some x and so is not in NoRepeats.

But then f is in Total iff g is in NoRepeat.

TOTAL $\leq_m$ NoRepeats as we were to show.

Assignment # 8.4 Key

4. Use reduction from NoRepeats to Total This, together with #3, shows NoRepeats and is Total are equally hard. Let f be an arbitrary natural numbers.

Let f be an arbitrary index.

Define g by $\varphi_g(\langle x,y \rangle) = \mu z\, [(x = y) \,||\, (\varphi_f(x) \mathrel{!=} \varphi_f(y))]$.

Clearly, $\varphi_g(\langle x,y \rangle) = 0$, for x,y, if $\varphi_f(x)\downarrow$ and $\varphi_f(y)\downarrow$, and either x=y or $\varphi_f(x) \mathrel{!=} \varphi_f(y)$

else for some x,y, $\varphi_g(\langle x,y \rangle)\uparrow$

Summarizing, f is in NoRepeats implies g is in Total and f is not in NoRepeats implies g diverges somewhere and so is not in Total.

NoRepeats $\leq_m$ Total as we were to show.

Assignment # 8.5 Key

5. Use Rice's Theorem to show that Non-Trivial is undecidable

First, Non-Trivial is non-trivial as S(x) = x+1 is in Non-Trivial and C0(x) = 0 is not.

Second, Non-Trivial is an I/O property.

To see this, let f and g are two arbitrary indices such that
$\forall x\ [\varphi_f(x) = \varphi_g(x)]$

f ∈ Non-Trivial iff ∃ y,z, y ≠ z, such that $\varphi_f(y)\downarrow$, $\varphi_f(z)\downarrow$ and $\varphi_f(y) \neq \varphi_f(z)$
iff, since $\forall x\ [\varphi_f(x) = \forall x\ \varphi_g(x)]$, ∃ y,z, y ≠ z, (same y,z as above) such that
$\varphi_g(y)\downarrow$, $\varphi_g(z)\downarrow$ and $\varphi_g(y) \neq \varphi_g(z)$ iff g ∈ Non-Trivial.

Thus, f ∈ Non-Trivial iff g ∈ Non-Trivial.

Assignment # 8.6 Key

6. **Use Rice's Theorem to show that NoRepeats is undecidable**

First, NoRepeats is non-trivial as S(x) = x+1 is in NoRepeats and C0(x) = 0 is not.

Second, NoRepeats is an I/O property.

To see this, let f and g are two arbitrary indices such that $\forall x\, [\varphi_f(x) = \varphi_g(x)]$.

f ∈ NoRepeats iff, for all x,y $\varphi_f(x)\downarrow$, $\varphi_f(y)\downarrow$ and $x \neq y \Rightarrow \varphi_f(x) \neq \varphi_f(y)$ iff, since $\forall x\, [\varphi_f(x) = \varphi_g(x)]$, for all x,y, $\varphi_g(x)\downarrow$, $\varphi_g(y)\downarrow$ and $x \neq y \Rightarrow \varphi_g(x) \neq \varphi_g(y)$ iff g ∈ NoRepeats.

Thus, f ∈ NoRepeats iff g ∈ NoRepeats.