# Assignment # 7.1a Sample Key

1. For the following languages, either provide a grammar to show it is a CFL or employ the Pumping Lemma to show it is not

   a.) **L = { $a^i b^j$ | j > 2*I }**

   *This language is a CFL. A grammar that works is*

   *$S \rightarrow aSbb \mid Sb \mid b$*

# Assignment # 7.1b Sample Key

1. b.) **L = { $a^n b^{n!}$ | n>0 }**

   *PL: Provides N>0*

   *We: Choose $a^N b^{N!} \in L$*

   *PL: Splits $a^N b^{N!}$ into uvwxy, $|vwx| \leq N$, $|vx| > 0$, such that $\forall i \geq 0\ uv^iwx^iy \in L$*

   *We: Choose i=2*

   *Case 1: vwx contains only b's, then we are increasing the number of b's while leaving the number of a's unchanged. In this case $uv^2wx^2y$ is of form $a^N b^{N!+c}$, c>0 and this is not in L.*

   *Case 2: vwx contains some a's and maybe some b's. Under this circumstances $uv^2wx^2y$ has at least N+1 a's and at most N!+N-1 b's. But $(N+1)! = N!(N+1) = N!*N+N \geq N! + N > N!+N-1$ and so is not in L.*

   *Cases 1 and 2 cover all possible situations, so L is not a CFL*

# Assignment # 7.2 Sample Key

2. Consider the context-free grammar **G = ( { S } , { a , b } , S , P )**, where **P** is:

   $\mathbf{S \rightarrow S\,a\,S\,b\,S \mid S\,b\,S\,a\,S \mid S\,a\,S\,a\,S \mid a \mid \lambda}$

   Provide the first part of the proof that

   **L ( G ) = L = { w | w** has at least as many **a's** as **b's }**

   That is, show that $\mathbf{L(G) \subseteq L}$

   To attack this problem we can first introduce the notation that, for a syntactic form $\alpha$, $\alpha_a$ = the number of **a's** in $\alpha$, and $\alpha_b$ = the number of **b's** in $\alpha$. Using this, we show that if $\mathbf{S} \Rightarrow^* \alpha$, then $\alpha_b \leq \alpha_a$ and hence that $\mathbf{L(G) \subseteq L}$:

   A straightforward approach is to show, inductively on the number of steps, **i**, in a derivation, that, if $\mathbf{S} \Rightarrow^{i} \alpha$, then $\alpha_b \leq \alpha_a$ .

# Assignment # 7.2 Sample Key

Basis (i=1): Since S $\Rightarrow \alpha$ iff S $\rightarrow \alpha$ and all rhs of S have $\alpha_b \leq \alpha_a$ then the base case holds

IH: Assume if S $\Rightarrow_m \alpha$ , then $\alpha_b \leq \alpha_a$ , whenever $m \leq n$

IS: Show that if S $\Rightarrow_{n+1} \alpha$ , then $\alpha_b \leq \alpha_a$

If S $\alpha$ then S $\Rightarrow_n \beta$ and $\beta \Rightarrow \alpha$

Since G has only one non-terminal S, the rewriting of $\beta$ to $\alpha$ involves a single application of one of the S-rules. By the I.H., $\beta$ has the property that $\beta_b \leq \beta_a$. Since a single application of an S rule either adds no b's or a's, one a, one a and one b, or two b's, we have the three following cases:

# Assignment # 7.2 Sample key

Case 0: $\alpha_a = \beta_a$, and $\alpha_b = \beta_b$

In which case, using the IH, we have:

$\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$

Case 1: $\alpha_b = \beta_b$, and $\alpha_a = \beta_a + 1$

In which case, using the IH, we have:

$\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$

Case 2: $\alpha_b = \beta_b + 1$, and $\alpha_a = \beta_a + 1$

In which case, using the IH, we have:

$\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$

Case 3: $\alpha_b = \beta_b$, and $\alpha_a = \beta_a + 2$

In which case, using the IH, we have:

$\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$