RICE'S THEOREM (S)

THERE ARE THREE (3) CONSTRAINTS

1. IT IS ABOUT PROPERTIES OF COMPUTABLE FUNCTIONS (PROGRAMS, PROCEDURES)
   IT SPLITS THE INDICES OF FUNCTIONS INTO TWO CLASSES. FOR SOME PROPERTY, P, WE HAVE SETS S_P AND $\overline{S_P}$
   $S_P = \{ f \mid \varphi_f$ HAS PROPERTY P $\}$
2. IT CAN BE APPLIED ONLY TO NON-TRIVIAL PROPERTIES. P IS NON-TRIVIAL IF
   $S_P \neq \phi$ <u>AND</u> $\overline{S_P} \neq \phi$
3. IT MUST BE A PROPERTY ABOUT FUNCTIONAL BEHAVIOR, NOT IMPLEMENTATION OR EFFICIENCY. THAT IS, IF f AND g HAVE SAME BEHAVIOR THEN BOTH ARE IN S_P OR BOTH ARE IN $\overline{S_P}$

RICE'S VARIANTS BASED ON BEHAVIOR TYPES

I. WEAK BEHAVIOR TYPE BASED ON DOMAIN

f AND g HAVE SAME BEHAVIOR IF
$Dom(f) = Dom(g)$ – STANDARD RICE

II. WEAK BEHAVIOR TYPE BASED ON RANGE

f AND g HAVE SAME BEHAVIOR IF
$Rng(f) = Rng(g)$

III. STRONG BEHAVIOR TYPE BASED ON ACTUAL MAPPINGS FROM INPUT TO OUTPUT

f AN g HAVE SAME BEHAVIOR IF
$\forall x\ f(x) = g(x)$
THIS MEANS SAME VALUE IF CONVERGE AND BOTH DIVERGE OTHERWISE

$\text{III} \Rightarrow \text{I}$ AND II
BUT $\text{I} \not\Rightarrow \text{III}$, $\text{II} \not\Rightarrow \text{III}$ AND
$\text{I} \,\&\, \text{II} \not\Rightarrow \text{III}$

RICE'S THEOREM STATEMENT

IF P IS A NON-TRIVIAL PROPERTY OF FUNCTION INDICES THAT IS IMMUNE TO IMPLEMENTATION (AN I/O PROPERTY) THEN P IS UNDECIDABLE (NON-RECURSIVE)

KEYS TO PROOF ARE:

1. <u>ALL</u> FUNCTIONS WITH EMPTY DOMAINS/ RANGES WILL EITHER HAVE PROPERTY P OR NOT HAVE PROPERTY P.
2. ASSUME ALL FUNCTIONS WITH EMPTY DOMAINS/RANGES ARE IN S_P. IF NOT WE PROVE THE COMPLEMENT OF P TO BE UNSOLVABLE
3. AS P IS NON-TRIVIAL, THERE IS SOME FUNCTION INDEX IN S_P. CALL THIS INDEX r

PROOF OF RICE'S THEOREM

LET P MEET ALL CONDITIONS OF RICE'S THEOREM AND LET $r \in S_P$. WE KNOW THAT $\emptyset \in \overline{S_P}$ WHERE $\emptyset$ INDEX IS OF SOME FUNCTION THAT DIVERGES EVERYWHERE

CONSIDER $HALT = \{\langle x,y \rangle \mid \varphi_x(y)\downarrow\}$

DEFINE $f_{r,x,y}$

r AS ABOVE
x, y ARBITRARY

AS
$$f_{r,x,y}(z) = \varphi_x(y) - \varphi_x(y) + \varphi_r(z)$$

IF $\langle x,y \rangle \in HALT$ THEN
$f_{r,x,y}(z) = \varphi_r(z)$ AND $f_{r,x,y} \in S_P$

IF $\langle x,y \rangle \notin HALT$ THEN
$\forall z \, f_{r,x,y}(z)\uparrow$ AND $f_{r,x,y} \in \overline{S_P}$

THUS,
$$HALT \leq S_P$$
AND SO P IS NON-RECURSIVE

RICE EXAMPLES (WEAK)

$\text{HasZero} = \{f \mid \exists x\ f(x)=0\}$

$C_0 \in \text{HasZero}$; $C_1 \notin \text{HasZero}$

SO NON-TRIVIAL

CAN USE VERSION II OF RICE

LET f AND g BE ARB. INDICES SUCH THAT

$\text{RNG}(f) = \text{RNG}(g)$

$f \in \text{HasZero}$ IFF $\exists x\ f(x)=0$

IFF $0 \in \text{RNG}(f)$

IFF $0 \in \text{RNG}(g)$

IFF $\exists x\ g(x)=0$

IFF $g \in \text{Has Zero}$

BY RICE'S HasZero IS NON-RECURSIVE

$\text{TOTAL} = \{f \mid \forall x\ f(x)\downarrow\}$

$C_0 \in \text{TOTAL}$ $\quad \mu y[y=y+1] \notin \text{TOTAL}$

SO NON-TRIVIAL

CAN USE VERSION I OF RICE

LET f AND g BE ARB. INDICES SUCH THAT

$\text{Dom}(f) = \text{Dom}(g)$

$f \in \text{TOTAL} \Leftrightarrow \forall x\ f(x)\downarrow \Leftrightarrow \text{Dom}(f) = \mathbb{N}$

$\Leftrightarrow \text{Dom}(g) = \mathbb{N} \Leftrightarrow \forall x\ g(x)\downarrow \Leftrightarrow g \in \text{TOTAL}$

BY RICE'S TOTAL IS NON-RECURSIVE

RICE EXAMPLE (STRONG)

$\text{MONOINCR} = \{f \mid \forall x\ f(x+1) > f(x)\}$

$S(x) = x+1 \in \text{MONOINCR}$

$C_0(x) = 0 \notin \text{MONOINCR}$

SO NON-TRIVIAL

MUST USE VERSION 3 OF RICE

LET f AND g BE ARB. INDICES SUCH THAT

$\forall x\ f(x) = g(x)$

$f \in \text{MONOINCR}$ IFF $\forall x\ f(x+1) > f(x)$

IFF $\forall x\ g(x+1) > g(x)$

AS $\forall x\ g(x) = f(x)$

IFF $g \in \text{MONOINCR}$

USEFUL FUNCTIONS (STP, VALUE)

WHEN DEVELOPING UNIVERSAL FUNCTION
WE CREATED A PREDICATE STP
TO SEE IF FUNCTION HALTS IN SOME
FIXED TIME FOR SOME FIXED INPUT

$STP(f, x, t) =$ TRUE
IFF $\varphi_f(x)$ CONVERGES IN t
OR FEWER STEPS

WE ALSO NEEDED A WAY TO EXTRACT
OUTPUT IF $\varphi_f(x)$ HALTS

$VALUE(f, x, t) = f(x)$
PROVIDED $STP(f, x, t)$
ELSE $VALUE(f, x, t)$ GIVES
A MEANINGLESS VALUE (WE USE 0)

USES OF STP AND VALUE

$\langle f, x \rangle \in HALT \Leftrightarrow \exists t\, [STP(f,x,t)]$

$f \in TOTAL \Leftrightarrow \forall x \exists t\, [STP(f,x,t)]$

$f \in HasZero \Leftrightarrow \exists \langle x,t \rangle [STP(f,x,t)\ \&\&$
$VALUE(f,x,t) == 0]$

$f \in MonoIncr \Leftrightarrow \forall x \exists t\, [STP(f,x,t)\ \&\&$
$STP(f,x+1,t)\ \&\&$
$VALUE(f,x,t) <$
$VALUE(f,x+1,t)]$

$f \in EMPTY \Leftrightarrow \forall \langle x,t \rangle [\,\overline{STP(f,x,t)}\,]$

CATEGORIES (P ALGORITHMIC PREDICATE)

$x \in S \Leftrightarrow P(x)$ RECURSIVE

$x \in S \Leftrightarrow \exists y\, P(x,y)$ RE

$x \in S \Leftrightarrow \forall y\, P(x,y)$ CO-RE

$x \in S \Leftrightarrow \forall y \exists z\, P(x,y,z)$ NON-RE, NON-CORE

$x \in S \Leftrightarrow \exists y \forall z\, P(x,y,z)$ NON-RE NON-CORE

RE ≡ SEMI DEC. (PART 1)

LET S BE RE THEN EITHER

$S = \emptyset$ OR $S = \mathrm{RNG}(f_S)$

WHERE f_S IS AN ALGORITHM

WE CAN BUILD g_S AS

$$g_S(x) = \mu y\,[y = y+1] \text{ IF } S = \emptyset$$

OR AS

$$g_S(x) = \mu y\,[f_S(y) = x]$$

OR

$$g_S(x) = \exists y\,[f_S(y) = x]$$

OR

$$g_S(x) = (\exists y\, f_S(y) = x) * x$$

IN ALL THREE CASES

$$\mathrm{Dom}(g_S) = S$$

IN THIRD CASE

$$\mathrm{Dom}(g_S) = \mathrm{RNG}(g_S) = S$$

THUS,

RE ⇒ SEMI DEC.

RE ≡ SEMI-DEC. (PART 2)

LET S BE SEMI-DECIDABLE THEN

$S = Dom(g_S)$

WHERE g_S IS A PROCEDURE

WE CAN BUILD f_S IF WE ASSUME

$S \neq \emptyset$

WE CAN DO SO SINCE IF $S = \emptyset$ THEN IT IS RE BY DEFINITION

LET $a \in S$ BE AN ARB. ELEMENT OF S

$$f_S(\langle x,t\rangle) = STP(g_S, x, t) * x + (1 - STP(g_S, x, t)) * a$$

f_S IS AN ALGORITHM

$RNG(f_S) = Dom(g_S)$

NOTE: f_S ENUMERATES EACH ELEMENT OF S INFINITELY OFTEN

RESULT IS

SEMI-DEC ⇒ RE

NOTE: f_S IS <u>PRIMITIVE</u> REC.

REC & RE

LET S BE A RECURSIVE (DECIDABLE) SET

S HAS AN ASSOCIATED ALGORITHMIC PREDICATE

χ_S, CALLED ITS CHARACTERISTIC FUNCTION SUCH THAT

$x \in S \iff \chi_S(x)$

THE COMPLEMENT OF S, $\bar{S}$, IS ALSO RECURSIVE AND

HAS IT'S OWN CHARACTERISTIC FUNCTION, $\chi_{\bar{S}}$

$x \in \bar{S} \iff \chi_{\bar{S}}(x) \iff \chi_S(x) = 0$

S REC $\Rightarrow$ S IS RE && $\bar{S}$ IS RE

CAN DEFINE g_S & $g_{\bar{S}}$ WHERE $S = Dom(g_S)$, $\bar{S} = Dom(g_{\bar{S}})$

BY

$g_S(x) = \mu y \, \chi_S(x)$ - DIVERGE IFF $\overline{\chi_S(x)}$

$g_{\bar{S}}(x) = \mu y \, \chi_{\bar{S}}(x)$ - DIVERGE IFF $\overline{\chi_{\bar{S}}(x)}$ IFF $\chi_S(x)$

S REC $\Leftrightarrow$ S & $\overline{S}$ ARE BOTH RE

ON PREVIOUS PAGE WE SHOWED

S REC $\Rightarrow$ S & $\overline{S}$ ARE BOTH RE

NEED

S RE & $\overline{S}$ RE $\Rightarrow$ S REC

S RE $\Leftrightarrow$ (i) $S=\emptyset$ OR (ii) $S=RNG(f_S)$, f_S ALG.

ALSO S RE $\Leftrightarrow$ $S = Dom(g_S)$, g_S A PROCEDURE

WE WILL USE $S=Dom(g_S)$, g_S A PROC.

AS $\overline{S}$ IS RE $\overline{S}=Dom(g_{\overline{S}})$, $g_{\overline{S}}$ A PROC.

$$\chi_S(x) = STP(g_S, x, \mu t[STP(g_S, x, t) \,||\, STP(g_{\overline{S}}, x, t)])$$

COULD DO OTHER WAY

IF $S=\emptyset$ THEN $\forall x\, \chi_S(x)=0$ } REC

IF $S=\mathbb{N}$ THEN $\forall x\, \chi_S(x)=1$

ASSUME $S\neq\emptyset$ AND $S\neq\mathbb{N}$ ($\overline{S}\neq\emptyset$)

AND $S=RNG(f_S)$ $\overline{S}=RNG(f_{\overline{S}})$

THEN

$$\chi_S(x) = f_S(\mu y[f_S(y)==x \,||\, f_{\overline{S}}(y)==x]) == x$$

GO TO PAGE 359