PROBLEM CATEGORIES

RECURSIVE (SOLVABLE)

LOTS OF EXAMPLES

RE, NON-RECURSIVE (UNDEC BUT SEMI DEC)

$HALT = \{ \langle f, x \rangle \mid \varphi_f(x)\downarrow \}$

SHOWN BY DIAGONALIZATION

NON-RE (CANNOT EVEN SEMI-DECIDE)

$TOTAL = \{ f \mid \forall x \ \varphi_f(x)\downarrow \}$

SHOWING COMPLEXITY OF NEW PROBLEMS

FIRST TECHNIQUE IS REDUCTION

LET B BE SOME SET OF UNKOWN COMPLEXITY

LET A BE SOME SET OF KNOWN COMPLEXITY

LET f BE A COMPUTABLE M-1 FUNCTION

$$A \leq_m B \text{ OR JUST } A \leq B$$

VIA f IF

$x \in A$ IFF $f(x) \in B$

IF A IS RE, NON-REC. THEN B IS NON-REC., BUT NOT NEC. RE

IF A IS NON-RE THEN B IS NON-RE AND, OF COURSE, NON-REC.

REDUCTION EXAMPLE #1

SHOW HALT $\leq$ TOTAL

LET f, x BE ARBITRARY NAT. NUMBERS

$\langle f,x \rangle \in$ HALT IFF $\varphi_f(x)\downarrow$

DEFINE F_x BY

$\forall y\, \varphi F_x(y) = \varphi f(x)$ // IGNORES INPUT

NOW

$\langle f,x \rangle \in$ HALT IFF $F_x \in$ TOTAL

THUS, HALT $\leq$ TOTAL

BY THIS, TOTAL IS NON-REC. BUT WE DO NOT KNOW IF IT'S RE

(WELL, WE DO, AND IT'S NOT)

NOTE: WE CAN LEAVE OUT φ AND JUST SAY

$\forall y\, F_x(y) = f(x)$ // OVERLOAD

FOR CONVENIENCE

EXAMPLE #2

$\text{HASZERO} = \{f \mid \exists x\ f(x) = 0\}$ //SKIP ϕ

SHOW HASZERO IS NON-REC.

LET f, x BE ARB.

DEFINE F_x BY

$\forall y\ F_x(y) = f(x) - f(x)$

CLEARLY, $\forall y F_x(y) = 0$ IF $f(x)\downarrow$

ELSE $\forall y\ F_x(y)\uparrow$

SO

$\langle f, x\rangle \in \text{HALT} \Leftrightarrow F_x \in \text{HASZERO}$

THUS, HASZERO IS NON-REC. SINCE

$\text{HALT} \leq \text{HASZERO}$

BUT IS HASZERO RE?

WELL IT IS AND WE WILL SHOW

THAT LATER

EXAMPLE #3

$$\text{ZERO} = \{ f \mid \forall x\ f(x) = 0 \}$$

SHOW ZERO IS NON-RE

NOTE PRIOR EXAMPLE SHOWED NON-REC !!

LET f BE ARB.

DEFINE

$$h(x) = f(x) - f(x)$$

NOW

$$\begin{aligned} f \in \text{TOTAL} \quad &\text{IFF} \quad \forall x\ f(x)\downarrow \\ &\text{IFF} \quad \forall x\ h(x) = 0 \\ &\text{IFF} \quad h \in \text{ZERO} \end{aligned}$$

THUS,

$$\text{TOTAL} \leq \text{ZERO}$$

AND SO ZERO IS NON-RE

TYPES OF REDUCTION

M-1 $\leq_m$

1-1 $\leq_1$

TURING (AKA ORACLE)
$\leq_t$

DEGREES ARE EQUIV. CLASSES

$\equiv_1$

$\equiv_m$

$\equiv_t$

ONE CLASS WE CARE ABOUT IS COMPLETE DEGREE (HIGHEST) AMONG RE SETS

RE COMPLETE

S IS RE-COMPLETE IFF

(1) S IS RE

(2) IF T IS RE THEN $T \leq S$

HALT (AKA K_0) IS RE-COMPLETE

LET A BE ARB RE SET THEN

$A = Dom\ \varphi_a$ FOR SOME INDEX a

HERE $A = W_a$ (ENUMERATION THEOREM)

$$x \in A \Leftrightarrow x \in Dom(\varphi_a) \Leftrightarrow \varphi_a(x)\downarrow \Leftrightarrow \langle a, x \rangle \in K_0$$

THUS

$A \leq K_0$ (REALLY $A \leq_1 K_0$)

K IS ALSO RE-COMPLETE

$K = \{f \mid \varphi_f(f)\downarrow\}$

LET f, x BE ARB.

DEFINE $\forall y \; F_x(y) = f(x)$

$\langle f, x\rangle \in HALT(K_0) \Leftrightarrow$

$x \in Dom(f) \Leftrightarrow$

$\forall y \; F_x(y)\downarrow$

$\Rightarrow F_x \in K$

$\langle f, x\rangle \notin HALT(K_0) \Leftrightarrow$

$x \notin Dom(f) \Leftrightarrow$

$\forall y \; F_x(y)\uparrow$

$\Rightarrow F_x \notin K$

THUS,

$K_0 \leq K$ (ACTUALLY $K \equiv K_0$)

BUT K IS OBVIOUSLY RE

AND SO K IS RE-COMPLETE

$RT = \{ f \mid \exists C \ni \forall x\ f(x)\downarrow$ IN AT MOST C STEPS $\}$

$\exists C \forall x\ f(x)\downarrow$ IN C STEPS
$|x| \leq c$

RICE'S

NOTES 339

BASIC IDEA IS THAT
NON-TRIVIAL PROPERTIES
ABOUT SETS OF INDICES
OF PROCEDURES ()
ARE UNDECIDABLE