EQUIVALENCE RELATIONS

REFLEXIVE, SYMMETRIC, TRANSITIVE

LET R BE AN EQUIV RELATION OVER U

R CREATES EQUIV. CLASSES THAT PARTITION U INTO NON-OVERLAPPING SUBSETS. ALL ELEMENTS OF U ARE IN SOME SUBSET AS xRx, $\forall x \in U$

THE INDEX OF R IS THE NUMBER OF PARTITIONS

R HAS FINITE INDEX IF R INDUCES A FINITE NUMBER OF PARTITIONS

IF R IS OVER STRINGS, E.G., OVER Σ^*

THEN R IS CALLED RIGHT INVARIANT IFF

$$xRy \Rightarrow \forall z \; xzRyz$$

DFA INDUCES A RIGHT INVARIANT

EQUIVALENCE RELATION WHOSE INDEX IS $|Q|$ SO LONG AS ALL STATES ARE REACHABLE I.E., $\forall q \in Q\ \exists x\ \delta^*(q_0, x) = q$ WHERE $\mathcal{Q} = (Q, \Sigma, \delta, q_0, F)$

INDUCED RELATION $R_{\mathcal{Q}}$ IS

$x R_{\mathcal{Q}} y$ IFF $\delta^*(q_0 x) = \delta^*(q_0, y)$

OBVIOUSLY $\forall z\, \delta^*(q_0, xz) = \delta^*(q_0, yz)$ WHENEVER $\delta^*(q_0, x) = \delta^*(q_0, y)$ AS $\mathcal{Q}$ IS DETERMINISTIC AND SO

$R_{\mathcal{Q}}$ IS RIGHT INVARIANT

State Compatibility

$\mathcal{Q} = (Q, \Sigma, \delta, q_0, F)$ is some DFA

p, q are compatible iff

$$\forall z \in \Sigma^* \quad \delta^*(p, z) \in F \iff \delta^*(q, z) \in F$$

This means they both accept or both reject z, and do so for all z, including $z = \lambda$.

If two states are compatible, we can merge them.

Approach to minimization

1. Remove any state that is not reachable from start state (q_0 here)
2. Discover and merge compatible states

Step 1 is done via DFS

Step 2 is easier if we discover in compatible states

p, q are incompatible iff

$$\exists z \in \Sigma^* \quad \delta^*(p, z) \in F \ \& \ \delta^*(q, z) \notin F$$
$$\text{or} \quad \delta^*(p, z) \notin F \ \& \ \delta^*(q, z) \in F$$

MINIMIZATION BY INCOMPATIBILITY

	a	b	c
>1	5	2	2
2	1	6	2
3	2	4	5
4	3	6	2
5	3	6	5
6	1	3	4

2	X				
3	2,5 2,4	X			
4	X	1,3	X		
5	X	1,3 6	X	2,5	
6	X	~~3,6 2,4~~	X	~~1,3 3,6 2,4~~	~~1,3 3,6 4,5~~
	1	2	3	4	5

1,3

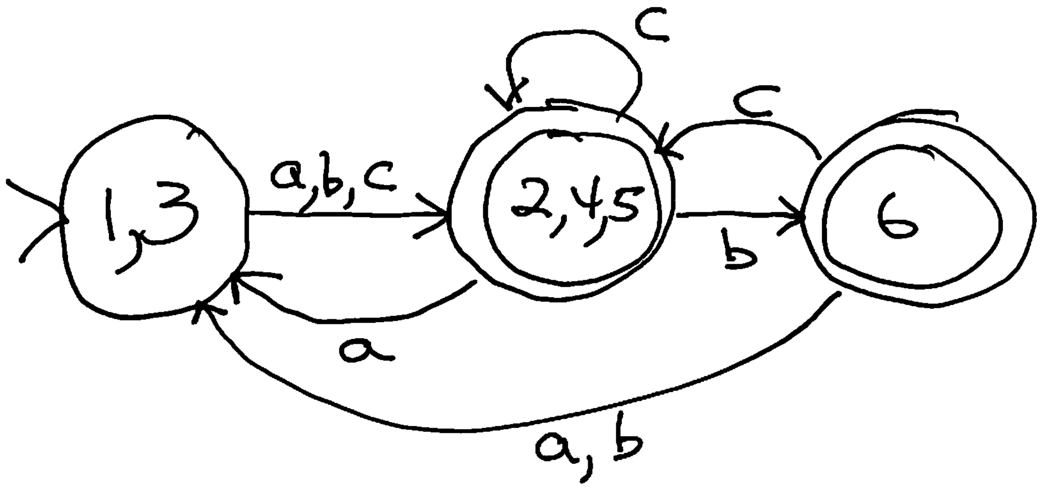
4,5

2,4

2,5

6

MINIMIZED DFA



NOTE: WE WILL PROVE THERE IS A UNIQUE MINIMUM ~~STATE~~ DFA FOR ANY REGULAR LANGUAGE L, UPTO RENAMING OF STATES

MYHILL - NERODE

THE FOLLOWING ARE EQUIVALENT

(1) $L = \mathcal{L}(Q)$ FOR SOME DFA Q

(2) L IS THE UNION OF SOME OF THE CLASSES OR A RIGHT INVARIANT EQUIV. REL., R, OF FINITE INDEX

(3) THE SPECIFIC RIGHT INVARIANT EQUIV. REL ASSOCIATED WITH L

$$xR_Ly \Leftrightarrow \forall z\, [xz \in L \Leftrightarrow yz \in L]$$

HAS FINITE INDEC

(1) $\Rightarrow$ (2). JUST USE R_Q FOR R

(2) $\Rightarrow$ (3) JUST NOTE

$$xRy \Rightarrow \forall z\,[xz\,R\,yz]$$
$$\Rightarrow \forall z\,[xz \in L \Leftrightarrow yz \in L]$$

AND SO R IS A REFINEMENT OF R_L AND HAS INDEX $\geq R_L$

THUS, INDEX OF R_L IS FINITE

(3) $\Rightarrow$ (1) JUST CHOOSE EACH PARTITION OF R_L AS A STATE AND

$$\delta([x], a) = [xa]$$

NOTION OF REPRESTATIVE ELEMENT

ON PRIOR PAGE WE DESCRIBED AN EQUIVALANCE CLASS OVER SOME SET OF STRINGS BY JUST CHOOSING A REPRESENTATIVE ELEMENT, SAY X, AND USING $[x]$ TO INDICATE $\{y \mid x R y\}$

NOTION OF UNIQUE MIN STATE DFA

STARTING WITH SOME DFA Q WE CAN USE TECHNIQUE SHOWN LAST TIME TO FIND Q', $Q = (Q, \Sigma, \delta, q_0, F)$, $Q' = (Q', \Sigma, \delta', q_0', F')$

WHERE $|Q'| \leq |Q|$ AND

$$x R_Q y \Rightarrow x R_{Q'} y$$

THIS IS PRECISELY THE AUTATAN YOU GET FROM MYHILL-NERODE'S STAGE 3

MYHILL-NERODE AND WHAT IS <u>NOT</u> REGULAR

FOR ANY LANGUAGE L, DEFINE R_L FROM MYHILL-NERODE, THAT IS,

$$xR_Ly \Leftrightarrow \forall z\,[xz \in L \Leftrightarrow yz \in L]$$

IF R_L HAS FINITE INDEX, L IS REGULAR,

ELSE L IS <u>NOT</u> REGULAR

E.G., LET $L = \{x \mid x \in \{0,1\}^*$ AND x IS ODD PARITY$\}$

TWO CLASSES INDUCED BY R_L ARE

$[\lambda] = \{x \mid x \in \{0,1\}^*$ AND x HAS EVEN PARITY$\}$

$[1] = \{x \mid x \in \{0,1\}^*$ AND x HAS ODD PARITY$\}$

$L = [1]$

SECOND POSITIVE EXAMPLE FROM M-N

DEFINE

$L = \{ x \mid x \in \{0,1\}^+ \text{ AND } x \bmod 5 = 1 \text{ OR } x \bmod 5 = 3 \}$

CLASSES (PARTITIONS ARE)

$[\lambda] = \{\lambda\}$

$[0] = \{ x \mid x \in \{0,1\}^+ \text{ AND } x \bmod 5 = 0 \}$

$[1] = \{ x \mid$ " " $x \bmod 5 = 1 \}$

$[2] = \{ x \mid$ " " $x \bmod 5 = 2 \}$

$[3] = \{ x \mid$ " " $x \bmod 5 = 3 \}$

$[4] = \{ x \mid$ " " $x \bmod 5 = 4 \}$

$L = [1] \cup [3]$

AND R_L HAS INDEX 6 WHICH IS FINITE

NOTE: THIS IS HOW I TEND TO NAME STATES OF DFAs

EXAMPLE WHERE M-N SHOWS L NON-REG.

$L = \{a^n b^n \mid n > 0\}$

NOTE! DO <u>NOT</u> NEED TO SHOW ALL CLASSES. IT IS SUFFICIENT TO SHOW AN INFINITE NUMBER OF PARTITIONS OF SOME EASILY SHOWN FORM

CONSIDER $[a^k b]_{R_L}$ $k > 0$

FOR EACH $i, j > 0$ AND $i \neq j$

CONSIDER $[a^i b]_{R_L}$ AND $[a^j b]_{R_L}$

$a^i b \cdot b^{i-1} \in L$

BUT

$a^j b \cdot b^{i-1} = a^j b^i, i \neq j, \notin L$

SO, FOR EACH DISTINCT $k > 0$, $[a^k b]_{R_L}$ IDENTIFIES A UNIQUE PARTITION, AND THUS THERE ARE AN INFINITE NUMER OF SUCH PARTITIONS AND SO R_L HAS INFINITE INDEX AND L IS NOT REGULAR.

M-N AGAIN

$L = \{xax \mid x \in \{a,b\}^+\}$

CONSIDER $[a^i b]_{R_L}$ $i > 0$

$a^i baa^i b \in L$, BUT

$a^k baa^i b \notin L$, $k < i$

AND SO $[a^i b]_{R_L} \neq [a^k b]_{R_L}$ $k < i$

THUS, $[a^i b]_{R_L}$ IS DISTINCT FOR EACH $i > 0$,

AS EACH $[a^i b]_{R_L}$ IS DISTINCT, $i > 0$,

R_L'S INDEX IS INFINITE AND

L IS NOT REGULAR

M-N YET AGAIN

$L = \{a^{FIB(k)} \mid k > 0\}$

CONSIDER $[a^{FIB(j)}]_{R_L}$, $j > 2$

IT'S CLEAR
$a^{FIB(j)} a^{FIB(j+1)} = a^{FIB(j)+FIB(j+1)}$

IS IN L

BUT $a^{FIB(k)} a^{FIB(j+1)} \notin L$ WHEN
$k > 2$, $k \neq j$, AND $k \neq j+2$

SO $[a^{FIB(j)}]_{R_L}$ IS UNIQUE FOR EACH $j > 2$

THIS MEANS INF. INDEX FOR R_L
AND SO L IS NOT REGULAR