

WEEK #3

①

1. REGULAR EXPRESSIONS

PRIMITIVE REG. EXPR. OVER Σ

EXPRESSION

SET

$\emptyset$
 λ

$\emptyset$
 $\{\lambda\}$

$a, a \in \Sigma$

$\{a\}$

CLOSURE OF EXPRESSIONS R, S DENOTING SETS R, S

$R + S$

$R \cup S = \{x \mid x \in R \text{ or } x \in S\}$

$R \cdot S$

$R \cdot S = \{xy \mid x \in R \text{ and } y \in S\}$

R^*

$R^* = \{\lambda\} \cup R \cup R^2 \cup \dots$

FOR CONVENIENCE WE ALSO ALLOW

R^+

$R^+ = R \cup R^2 \cup R^3 \cup \dots$

PRECEDENCE

$* > \cdot > +$

$\cdot$ AND $+$ ARE ASSOCIATED LEFT TO RIGHT

$*$ IS NON-ASSOCIATIVE OR JUST REALIZE $(R^*)^* = R^*$

PARENTHESES ARE USED TO OVERRIDE

DEFAULT PARSING OR FOR READABILITY

2. REGULAR SETS

EVERY REGULAR EXPRESSION
DENOTES A REGULAR SET

2. EVERY REGULAR SET IS A REGULAR LANG.

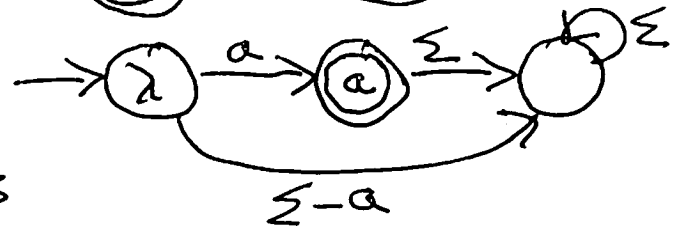
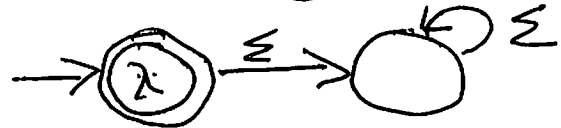
REGULAR EXPRESSION

$\emptyset$

λ

a

DFA



OR COULD DO NFA'S

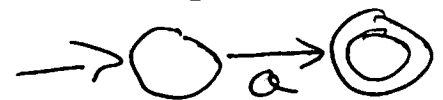
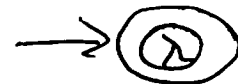
REG EXP

$\emptyset$

λ

a

NFA

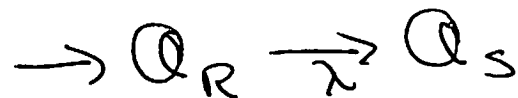
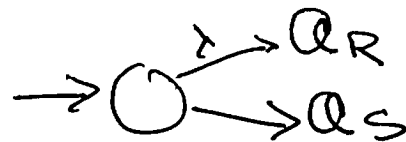


AS REGULAR LANGUAGES CLOSED UNDER $\cup, \circ, *$

THEN GET ALL REG. SETS AS
REG. LANGUAGES

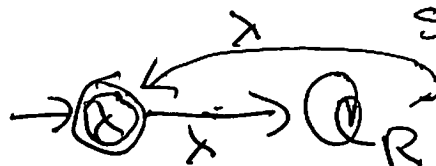
$R + S$

$R \circ S$



FROM EACH FINAL
STATE OF R TO
START STATE OF S

R^*



WEEK #3

③

3. EVERY REGULAR LANGUAGE IS A REG. SET

Q. APPROACH #1: $R_{i,j}^k$ SETS (REG. EXPR.)

LET $L = \mathcal{L}(Q)$ WHERE

$$Q = (Q, \Sigma, \delta, q_1, F), Q = \{q_1, \dots, q_n\}$$

WANT TO BUILD EXPRESSIONS

$R_{i,j}^k$, WHERE $1 \leq i, j \leq n, 0 \leq k \leq n$

SEMANTICALLY,

$$R_{i,j}^k = \{w \mid \delta^*(q_i, w) = q_j \text{ AND} \\ \text{PATH FROM } q_i \text{ TO } q_j \text{ INVOLVES} \\ \text{NO STATE WITH INDEX } > k\}$$

ACTUALLY $R_{i,j}^k$ IS REG. EXPR FOR THIS SET

$$\begin{aligned} \text{BASIS: } R_{i,j}^0 &= \emptyset \text{ IF } i \neq j \text{ AND } \delta(q_i, a) \neq q_j, \forall a \in \Sigma \\ R_{i,j}^0 &= a \text{ IF } \delta(q_i, a) = q_j \text{ AND } i \neq j \\ R_{i,i}^0 &= \lambda \text{ IF } i = j \text{ AND } \delta(q_i, a) \neq q_i, \forall a \in \Sigma \\ R_{i,i}^0 &= \lambda + a \text{ IF } i = j \text{ AND } \delta(q_i, a) = q_i \end{aligned}$$

INDUCTIVE HYP: ASSUME $R_{i,j}^m$ REG EXPR (SET)
 $0 \leq m \leq k, 1 \leq i, j \leq n$

INDUCTIVE STEP:

$$R_{i,j}^{k+1} = R_{i,j}^k + R_{i,k+1}^k (R_{k+1,k+1}^k)^* R_{k+1,j}^k$$

$$\mathcal{L}(Q) = \bigcup_{q_1 \in F} R_{1,q_1}^n$$

[EXAMPLE ON 8/8/88]

3. b. STATE RIPPING

LET $L = \mathcal{L}(Q)$ WHERE

$$Q = (Q, \Sigma, \delta, q_i, F), Q = \{q_1, \dots, q_n\}$$

CONSTRUCT

$$Q' = (Q \cup \{q_0, q_s\}, \Sigma, \delta', q_0, \{q_s\}) \quad q_0, q_s \in Q$$

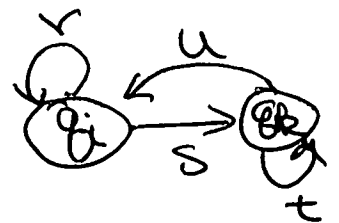
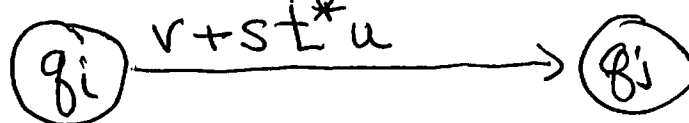
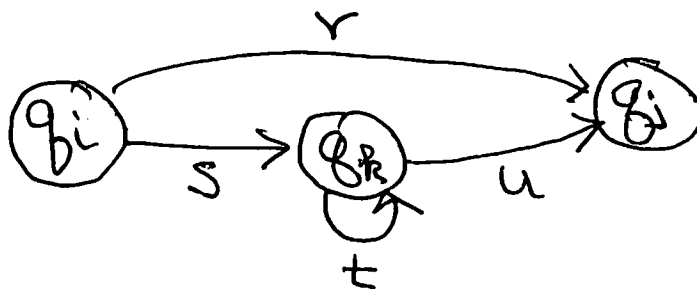
$$\delta'(q_0, \lambda) = \{q_i\} \quad \delta'(q_i, \lambda) = \{q_s\} \quad \forall q_i \in F$$

r_i INITIALLY IS REG. EXP. THAT DIRECTLY GOES FROM q_i TO q_s .
WE WILL COLLAPSE STATES IN Q , EXCEPT

FOR q_0 & q_s . AS WE RIP A STATE q_R OUT, WE WILL EXTEND EXPRESSION r_i BY ARCS THAT LED INTO q_R FROM q_i AND, TRANSITIVELY TO q_i , BY AN EXPR.

THAT REPRESENTS $r_i \cdot r_R^* \cdot r_{Ri} + r_{Li}$

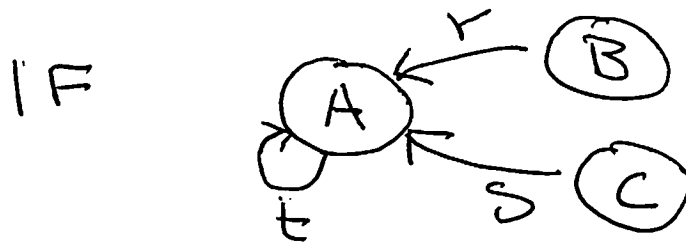
IN END $\mathcal{L}(Q) = r_0 f$



OF COURSE MUST DO FOR ALL PAIRS

[EXAMPLE ON 92-94]

3.C. REGULAR EQUATIONS



THEN $A = Br + Cs + At$

IF A IS START STATE THEN

$$A = \lambda + Br + Cs + At$$

OF COURSE, THERE IS A TERM FOR EVERY STATE THAT HAS AN ARC LEADING INTO A.

DO THIS FOR ALL STATES AND HAVE r EQUATIONS IN n UNKNOWN S THAT SEEMS EASY BUT SOME MAY BE RECURSIVE IF AUTOMATA HAS LOOPS

WE CLAIM THAT IF $\lambda \notin P$ THEN

$$R = Q + RP$$

HAS UNIQUE SOLUTION

$$R = QP^*$$

3 c. CONTINUED

(1) QP^* IS A SOLUTION

$$R = Q + RP = Q + QP^*P = Q + QP^+ = QP^*$$

(2) ANY SOLUTION IS CONTAINED IN QP^*

$$\begin{aligned} R &= Q + RP = Q + (Q + RP)P = Q + QP + RP^2 \\ &= Q + QP + (Q + RP)P^2 = Q + QP + QP^2 + RP^3 \\ &\dots \\ &= Q + QP + QP^2 + QP^3 + \dots + QP^k + RP^{k+1}, k \geq 0 \\ &= Q(\lambda + P + P^2 + \dots + P^k) + RP^{k+1}, k \geq 0 \end{aligned}$$

SINCE $\lambda \notin P$ THEN FOR ANY $w \in R, |w| = k,$
 $w \notin RP^{k+1}$ AND SO

$$w \in Q(\lambda + P + P^2 + \dots + P^k) \subseteq QP^*$$

$$\text{THUS } R \subseteq QP^*$$

(3) AS $QP^* \subseteq R$ AND $R \subseteq QP^*$

$$\text{THEN } R = QP^*$$

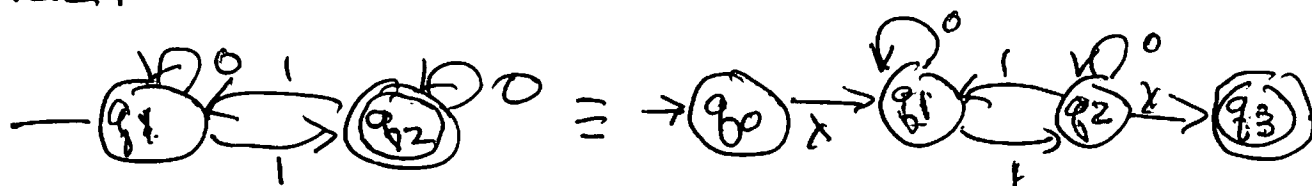
AGAIN PROVIDED $\lambda \notin P$

NOTE! WE WOULD ONLY SEE $\lambda \in P$ IF
 WE HAD $(R)^\perp$

WHICH IS USELESS

[EXAMPLES OF 98/99]

4. REALLY SIMPLE EXAMPLE



$$R_{11}^0 = \lambda + 0 \quad R_{22}^0 = 0$$

$$R_{12}^0 = 1 \quad R_{21}^0 = 1$$



$$R_{11}^1 = (\lambda + 0) + (\lambda + 0)(\lambda + 0)^*(\lambda + 0) = 0^*$$

$$R_{12}^1 = 1 + (\lambda + 0)(\lambda + 0)^*1 = 0^*1$$

$$R_{22}^1 = 0 + 1(\lambda + 0)^*1 = 0 + 10^*1$$

$$R_{21}^1 = 1 + 1(\lambda + 0)^*(\lambda + 0) = 10^*$$

$$R_{12}^2 = 0^*1 + 0^*1(0 + 10^*1)^*(0 + 10^*1)$$

$$= 0^*1(0 + 10^*1)^*$$

$$Q_1 = \lambda + Q_21 + Q_10$$

$$Q_2 = Q_11 + Q_20$$

$$Q_1 = (\lambda + Q_21)0^*$$

$$Q_2 = (\lambda + Q_21)0^*1 + Q_20$$

$$= 0^*1 + Q_2(0 + 10^*1)$$

$$= 0^*1(0 + 10^*1)^*$$

VERY RARE THAT ALL THREE
REG EXPR IN SAME FORM. NOTES HAVE
BETTER EXAMPLE.

WEEK #3

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WHAT COULD HAPPEN IF ALLOWED
 λ -TRANSITIONS IN REG. EQ.

APPROACH

CONSIDER $\rightarrow$ OVER $\Sigma = \{a\}$

$R = \lambda$ IS ONLY EQUATION
AND ALL IS WELL

NOW CONSIDER $\rightarrow$ OVER $\Sigma = \{a\}$

$$R = \lambda + R \cdot \lambda$$

CLEARLY $R = \lambda$ IS A SOLUTION

$$R = \lambda + \lambda \cdot \lambda = \lambda$$

BUT SO IS $R = \lambda + A$, WHERE A IS
ANY LANGUAGE OVER $\{a\}$

$$R = \lambda + (\lambda + A) \cdot \lambda = \lambda + A$$

BUT THEN THERE ARE AN
UNCOUNTABLY INFINITE NUMBER OF SOLUTIONS!

5. STATES AS EQUIV. CLASSES

NOTION OF STATES AS EQUIVALENCE CLASSES

LET $Q = (Q, \Sigma, \delta, q_0, F)$ BE SOME DFA

ASSUME EVERY STATE IN Q IS REACHABLE,

I.E., $\forall q \in Q \exists w \in \Sigma^* \Rightarrow \delta^*(q_0, w) = q$

CLEARLY, IF $\delta^*(q_0, x) = \delta^*(q_0, y)$

THEN $\forall z \delta^*(q_0, xz) = \delta^*(q_0, yz)$

DEFINE R_Q BY

$x R_Q y$ IFF $\delta^*(q_0, x) = \delta^*(q_0, y)$

THEN R_Q IS AN EQUIV. REL. THAT

PARTITIONS Σ^* INTO $|Q|$ EQUIV. CLASSES

AND THE UNION OF SOME OF THESE

CLASSES DEFINES $L(Q)$ --

CLASSES ASSOC. WITH F

6. MINIMIZATION OF DFA's

DEFINE STATES $p, q \in Q$ AS INDISTINGUISHABLE

$$\text{IF } \forall z [\delta^*(p, z) \in F \Leftrightarrow \delta^*(q, z) \in F]$$

IF TWO STATES ARE INDISTINGUISHABLE THEN THEY HAVE SAME FUTURES AND CAN BE MERGED.

MERGING STATES CREATES A NEW, LESS REFINED, EQUIV. REL. OVER Σ^* THAT STILL HAS PROPERTY THAT $\mathcal{I}(Q)$ IS UNION OF SOME CLASSES, BUT NUMBER OF CLASSES IS SMALLER.

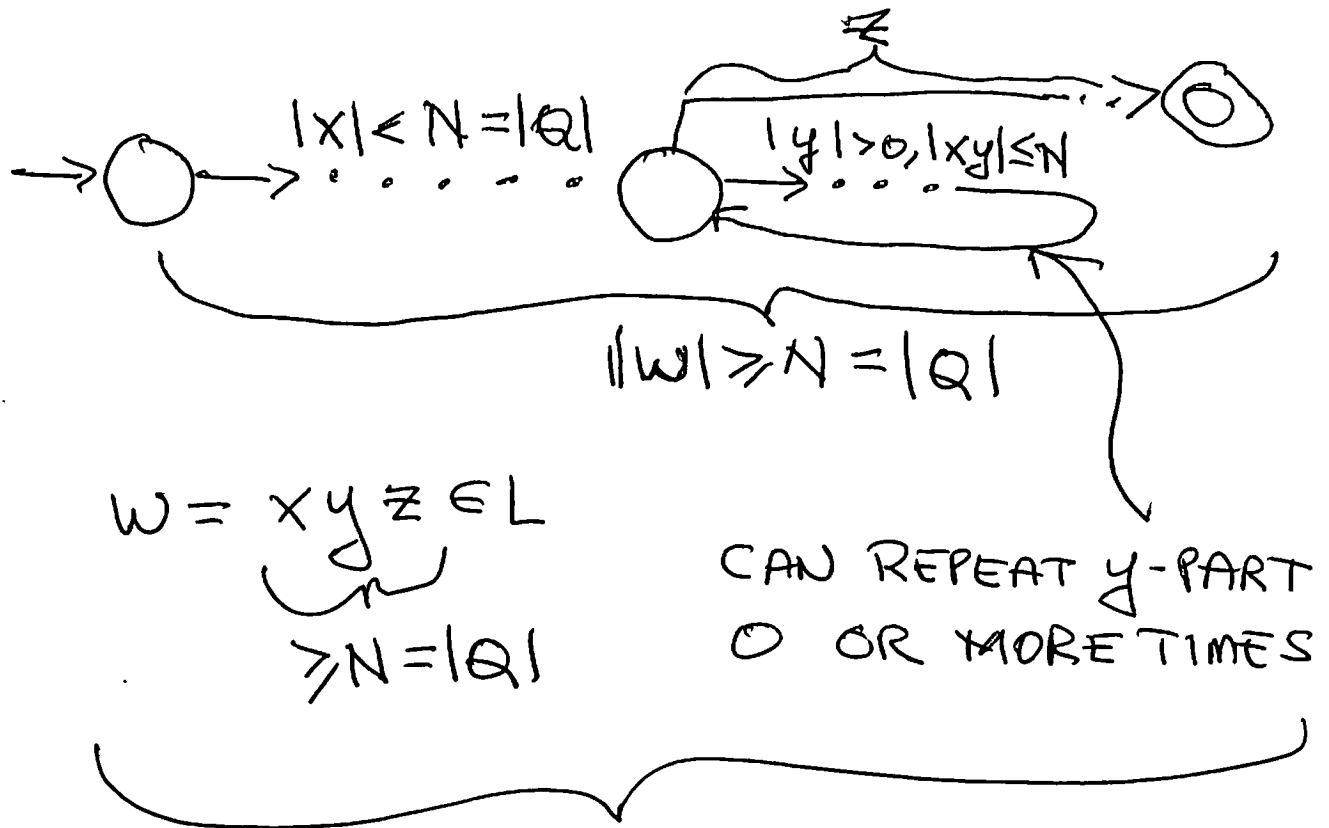
NEW FINAL STATES ARE ASSOCIATED WITH EQUIV. CLASSES CONTAINING STRINGS THAT LED TO FINAL STATES IN Q .

WE ACTUALLY ATTACK MINIMIZATION BY DISCOVERING DISTINGUISHABLE STATES

THE EASIEST APPROACH IS VIA A LOWER TRIANGULAR MATRIX

[GO TO ~~110/11~~] 100-101

7. PUMPING LEMMA



$$w = xy^i z \in L$$

$\underbrace{\quad}_{\geq N = |Q|}$

CAN REPEAT y -PART
0 OR MORE TIMES

CONSEQUENCE

$$xy^i z \in L, \forall i \geq 0$$

P.L. IS BASED ON PIGEONHOLE PRINCIPLE

IF HAVE N PLACES AND VISIT

THESE $> N$ TIMES THEN MUST

REUSE AT LEAST ONE

STRING OF LENGTH N MAKES $N+1$ VISITS

$$a_1, a_2, \dots, a_N$$