Assignment # 8.1 Sample Key

1. Use reduction from **HALT** to show that one cannot decide **HasExp**, where
   HasExp = { f | for some x, y, x<y, $\varphi_f(y) = 2^\wedge\varphi_f(x)$ }

Let f,x be an arbitrary pair of natural numbers. <f,x> ∈ Halt iff $\varphi_f(x)\downarrow$

Define g as index of φ_g where $\forall y\ \varphi_g(y) = \varphi_f(x) - \varphi_f(x) + y$

Clearly, $\forall y\ \varphi_g(y) = y$ iff $\varphi_f(x)\downarrow$; otherwise $\forall y\ \varphi_g(y)\uparrow$

But then, $\varphi_f(x)\downarrow$ iff $\forall y\ \varphi_g(y) = y$ and $\varphi_g(2^y) = 2^y$ implies $\varphi_g(0) = 0$ and $\varphi_g(1) = 2^0 = 1$ and so for some x, y, x<y, $\varphi_f(y) = 2^\wedge\varphi_f(x)$

Summarizing, <f,x> ∈ Halt iff g ∈ HasExp and so

Halt $\leq_m$ HasExp as we were to show.

Note: I have not overloaded the index of a function with the function in my proof, but I do not mind if you do such overloading.

Assignment # 8.2 Sample Key

2. Show that HasExp reduces to Halt. (1 plus 2 show they are equally hard)

Let f be an arbitrary natural number. $f \in$ HasExp iff for some x and y, $x < y$, $\varphi_f(x)\downarrow$, $\varphi_f(y)\downarrow$ and $\varphi_f(y) = 2 \wedge \varphi_f(x)$

Define g as index of φ_g where $\forall z\ \varphi_g(z) =$
$\exists$ <x,y,t> [STP(f,x,t) & STP(f,y,t) & (x<y) & (VALUE(f,x,t) = 2^VALUE(f,y,t))]

Clearly, $\forall z\ \varphi_g(z) = 1$, iff there is some pair, x,y, such that $x < y$ and $\varphi_f(y) = 2\wedge\varphi_f(x)$; and $\forall z\ \varphi_g(z)\uparrow$, otherwise

Summarizing, $f \in$ HasExp iff <g,0> $\in$ Halt and so

HasExp $\leq_m$ Halt as we were to show.

Assignment # 8.3 Sample Key

3. Use Reduction from TOTAL to show that IsExp is not even re, where
   IsExp = { f | for all x, there is some y, x<y, $\varphi_f(y) > 2^\wedge\varphi_f(x)$ }
   Note: If you use $\varphi_f(y) = 2^\wedge\varphi_f(x)$, that's okay

Let f be an index of some arbitrary function.
Define g as index of φ_g where $\forall x\ \varphi_g(x) = \varphi_f(x) - \varphi_f(x) + x$

Clearly, $\forall x\ \varphi_g(x) = x$, iff $\forall x\ \varphi_f(x)\downarrow$, and $\forall x\ \varphi_g(x)\uparrow$, otherwise.

But then, $\forall x\ \varphi_f(x)\downarrow$ iff $\forall x\ \varphi_g(x) = x$ and $\varphi_g(2^x+1) = 2^x+1$ (Here, y is 2^x+1)

Summarizing, f $\in$ TOTAL iff g $\in$ IsExp and so

TOTAL $\leq_m$ IsExp as we were to show.

Assignment # 8.4 Sample Key

4. Show IsExp reduces to TOTAL. (3 plus 4 show they are equally hard)

Let f be an arbitrary natural number. f is in IsExp iff
$\forall x \exists y, x<y, \varphi_f(y) > 2^\wedge\varphi_f(x)$.
Note: To be in IsExp, f must be in TOTAL since the property is true of all x.

Define g as index of φ_g where $\varphi_g(x) = \exists y\ [x<y\ \&\ \varphi_f(y) > 2^\wedge\varphi_f(x)]$

Clearly, $\forall x\ \varphi_g(x)\downarrow$ iff
$\forall x \exists y[\ y>x\ \&\ \varphi_f(x)\downarrow\ \&\ \varphi_f(y)\downarrow\ \&\ \varphi_f(y) > 2^\wedge\varphi_f(x)]$;
otherwise $\exists x\ \varphi_g(x)\uparrow$.

Summarizing, $f \in$ IsExp iff $g \in$ TOTAL and so

IsExp $\leq_m$ TOTAL as we were to show.

Assignment # 8.5 Sample Key

5. Use Rice's Theorem to show that HasExp is undecidable

First, IsExp is non-trivial as I(x) = x is in HasExp (for any x there is a $y=2^x > x$, such that $I(y) = 2^x$) and C0(x) = 0 is not.

Second, HasExp is an I/O property.

To see this, let f and g are two arbitrary indices such that $\forall x\ [\varphi_f(x) = \varphi_g(x)]$

**$f \in$ HasExp iff $\exists\ y,z\ [y<z\ \&\ \varphi_f(y)\downarrow\ \&\ \varphi_f(z)\downarrow\ \&\ \varphi_f(z) = 2^\wedge\varphi_f(y)$
iff, since $\forall x\ [\varphi_f(x) = \varphi_g(x)]$, $\exists\ y,z$ [y z, (same y,z as above) &
$\varphi_g(y)\downarrow\ \&\ \varphi_g(z)\downarrow\ \&\ \varphi_g(z) = 2^\wedge\varphi_g(y)$ iff $g \in$ HasExp**

Thus, $f \in$ HasExp iff $g \in$ HasExp.

Assignment # 8.6 Sample Key

6. Use Rice's Theorem to show that IsExp is undecidable

First, IsExp is non-trivial as I(x) = x is in IsExp (for every x, there is a $y=2^x+1$, such that I(y)>I(x)) and C0(x) = 0 is not.

Second, IsExp is an I/O property.

To see this, let f and g are two arbitrary indices such that $\forall x\ [\varphi_f(x) = \varphi_g(x)]$.

f $\in$ IsExp iff $\forall x \exists y\ [x < y,\ \varphi_f(x)\downarrow,\ \varphi_f(y)\downarrow$ and $\varphi_f(y) > 2^\wedge\varphi_f(x)$
iff, since $\forall x\ [\varphi_f(x) = \varphi_g(x)]$, $\forall x \exists y\ [\varphi_g(x)\downarrow,\ \varphi_g(y)\downarrow$ && $\varphi_g(y)>2^\wedge\varphi_g(x)$
iff g $\in$ IsExp.

Thus, f $\in$ IsExp iff g $\in$ IsExp.