1. Use reduction from **HALT** to show that one cannot decide **HasExp**, where

$$\text{HasExp} = \{ f \mid \text{for some } x, y, x < y, \varphi_f(y) = 2^{\varphi_f(x)} \}$$

Let \( f, x \) be an arbitrary pair of natural numbers. \( <f, x> \in \text{Halt} \iff \varphi_f(x) \downarrow \)

Define \( g \) as index of \( \varphi_g \) where \( \forall y \varphi_g(y) = \varphi_f(x) - \varphi_f(x) + y \)

Clearly, \( \forall y \varphi_g(y) = y \iff \varphi_f(x) \downarrow \); otherwise \( \forall y \varphi_g(y) \uparrow \)

But then, \( \varphi_f(x) \downarrow \iff \forall y \varphi_g(y) = y \) and \( \varphi_g(2^y) = 2^y \)

implies \( \varphi_g(0) = 0 \) and \( \varphi_g(1) = 2^0 = 1 \) and so for some \( x, y, x < y, \varphi_f(y) = 2^{\varphi_f(x)} \)

Summarizing, \( <f, x> \in \text{Halt} \iff g \in \text{HasExp} \) and so

**Halt \leq_m \text{HasExp}** as we were to show.

Note: I have not overloaded the index of a function with the function in my proof, but I do not mind if you do such overloading.
2. Show that HasExp reduces to Halt. (1 plus 2 show they are equally hard)

Let $f$ be an arbitrary natural number. $f \in \text{HasExp}$ iff for some $x$ and $y$, $x < y$, $\varphi_f(x) \downarrow$, $\varphi_f(y) \downarrow$ and $\varphi_f(y) = 2^{\varphi_f(x)}$

Define $g$ as index of $\varphi_g$ where $\forall z \varphi_g(z) =$

$\exists <x,y,t>$ $[\text{STP}(f,x,t) \& \text{STP}(f,y,t) \& (x < y) \& (\text{VALUE}(f,x,t) = 2^{\text{VALUE}(f,y,t)})]$

Clearly, $\forall z \varphi_g(z) = 1$, iff there is some pair, $x,y$, such that $x < y$ and $\varphi_f(y) = 2^{\varphi_f(x)}$; and $\forall z \varphi_g(z) \uparrow$, otherwise

Summarizing, $f \in \text{HasExp}$ iff $<g,0> \in \text{Halt}$ and so

\text{HasExp} \leq_m \text{Halt} \text{ as we were to show.}
3. Use Reduction from TOTAL to show that IsExp is not even re, where
IsExp = \{ f \mid \text{for all } x, \text{ there is some } y, x < y, \varphi_f(y) > 2^\varphi_f(x) \} 
Note: If you use \( \varphi_f(y) = 2^\varphi_f(x) \), that’s okay

Let \( f \) be an index of some arbitrary function.
Define \( g \) as index of \( \varphi_g \) where \( \forall x \varphi_g(x) = \varphi_f(x) - \varphi_f(x) + x \)
Clearly, \( \forall x \varphi_g(x) = x \), iff \( \forall x \varphi_f(x) \downarrow \), and \( \forall x \varphi_g(x) \uparrow \), otherwise.
But then, \( \forall x \varphi_f(x) \downarrow \) iff \( \forall x \varphi_g(x) = x \) and \( \varphi_g(2^x+1) = 2^x+1 \) (Here, \( y \) is \( 2^x+1 \))
Summarizing, \( f \in TOTAL \) iff \( g \in IsExp \) and so
TOTAL \( \leq_m \) IsExp as we were to show.
4. Show $\text{IsExp}$ reduces to $\text{TOTAL}$. (3 plus 4 show they are equally hard)

Let $f$ be an arbitrary natural number. $f$ is in $\text{IsExp}$ iff

$$\forall x \exists y, x < y, \varphi_f(y) > 2^\varphi_f(x).$$

Note: To be in $\text{IsExp}$, $f$ must be in $\text{TOTAL}$ since the property is true of all $x$.

Define $g$ as index of $\varphi_g$ where $\varphi_g(x) = \exists y [x < y \& \varphi_f(y) > 2^\varphi_f(x)]$

Clearly, $\forall x \varphi_g(x) \downarrow$ iff

$$\forall x \exists y [x > y \& \varphi_f(x) \downarrow \& \varphi_f(y) \downarrow \& \varphi_f(y) > 2^\varphi_f(x)];$$

otherwise $\exists x \varphi_g(x) \uparrow$.

Summarizing, $f \in \text{IsExp}$ iff $g \in \text{TOTAL}$ and so

$\text{IsExp} \leq_m \text{TOTAL}$ as we were to show.
Assignment # 8.5 Sample Key

5. Use Rice’s Theorem to show that HasExp is undecidable

First, IsExp is non-trivial as \( I(x) = x \) is in HasExp (for any \( x \) there is a \( y = 2^x > x \), such that \( I(y) = 2^x \)) and \( C0(x) = 0 \) is not.

Second, HasExp is an I/O property.

To see this, let \( f \) and \( g \) are two arbitrary indices such that

\[
\forall x \ [\phi_f(x) = \phi_g(x)]
\]

\( f \in \text{HasExp} \iff \exists y, z \ [y < z \land \phi_f(y) \downarrow \land \phi_f(z) \downarrow \land \phi_f(z) = 2^\phi_f(y) \]

iff, since \( \forall x \ [\phi_f(x) = \phi_g(x)] \), \( \exists y, z \ [y, z \text{ (same } y, z \text{ as above)} \land \phi_g(y) \downarrow \land \phi_g(z) \downarrow \land \phi_g(z) = 2^\phi_g(y) \iff g \in \text{HasExp} \)

Thus, \( f \in \text{HasExp} \iff g \in \text{HasExp} \).
6. Use Rice’s Theorem to show that \textit{IsExp} is undecidable

First, \textit{IsExp} is non-trivial as \(I(x) = x\) is in \textit{IsExp} (for every \(x\), there is a \(y = 2^x + 1\), such that \(I(y) > I(x)\)) and \(C0(x) = 0\) is not.

Second, \textit{IsExp} is an I/O property.

To see this, let \(f\) and \(g\) are two arbitrary indices such that

\[
\forall x \left[ \varphi_f(x) = \varphi_g(x) \right].
\]

\(f \in \text{IsExp} \iff \forall x \exists y \left[ x < y, \varphi_f(x) \downarrow, \varphi_f(y) \downarrow \text{ and } \varphi_f(y) > 2^{\varphi_f(x)} \right]
\]

iff, since \(\forall x \left[ \varphi_f(x) = \varphi_g(x) \right], \forall x \exists y \left[ \varphi_g(x) \downarrow, \varphi_g(y) \downarrow \text{ and } \varphi_g(y) > 2^{\varphi_g(x)} \right]
\]

iff \(g \in \text{IsExp} \).

Thus, \(f \in \text{IsExp} \iff g \in \text{IsExp} \).