Assignment # 7.1a Key

1. Write a CFG to show the language is a CFL or use the Pumping Lemma for CFLs to prove that it is not for each of the following.
   a) **L = { a^i b^j | j = i^2, i, j >0 }**
   Assume this language is a CFL
   PL: Provides N>0
   Me: a^N b^{N2}
   PL: a^N b^{N2} = uvwxy, |vwx| ≤ N, |vx|>0, and ∀i uv^iwx^iy ∈ L
   ME: i=2.
   Case 1) vwx contains some a's. Then uv^2wx^2y has at least N+1 a's and at most N^2 + N - 1 b's. But $(N+1)^2$ is N^2 + 2N + 1 and N^2 + 2N + 1 > N^2 + N - 1 since 2N+1 > N-1 for N>0 (even for N = 0), Thus, uv^2wx^2y ∉ L
   Case 2) vwx contains only b's. Then uv^2wx^2y has exactly N a's and at least N^2 + 1 b's. But then it has too many b's. Thus, uv^2wx^2y ∉ L
   These cases cover all possibilities, so L is not a CFL.

Assignment # 7.1b Key

1. Write a CFG to show the language is a CFL or use the Pumping Lemma for CFLs to prove that it is not for each of the following.

   b) **$L = \{ a^i\, b^j\, c^k\, d^m \mid m + k = i + j \}$**

   The language is a CFL as can be seen by G = ({S, }, {a,b,c,d}, R, S)

   S → a S d | T | U
   T → a T c | V
   U → b U d | V
   V → b V c | λ

Assignment # 7.2.1 Key

2. Consider the context-free grammar **G = { {S}, {a,b}, R, S }**
   R:
   S → a | b | a a | b b | a S a | b S b

Provide a proof that shows

L = { w | w ∈ {a,b}$^+$ and w is a palindrome }

You will need to provide an inductive proof in both directions. Actually, though, the best approach is to prove a Lemma first that is used in the proofs going in each direction of containment.

Lemma 1: **S ⇒* β**, where **β** contains a nonterminal, iff **β** is of the form **xSx^R**, where **x∈{a,b}***. We provide this inductively on the length of the derivation, i.e., we show, for all **k≥0** that **S $\Rightarrow^k$ xSx^R** for all strings **x∈{a,b}***, **|x|=k**, and that these are the only non-terminal strings that can be derived in **G** by derivations of length **k**.

Base (k=0): **S $\Rightarrow^0$ S** is the only length zero derivation. The form of this is **xSx** for **|x| = 0**, and that is the only string that can be derived in zero steps, so our base case is shown.

IH(k): S $\Rightarrow^k$ β, **k≥0**, where **β** contains a nonterminal, iff **β** is of the form **xSx^R**, where **x∈{a,b}* and |x|=k**.

IS(k+1): Each derivation of length **k+1** containing a non-terminal string in **L(G)** must start with application of one the following two rules **S → a S a | b S b**. This means that any derivation of length k+1 starts **S ⇒ aSa** or **S ⇒ bSb** followed by a derivation from **S** of length **k**. By IH, we have either **S ⇒ aSa $\Rightarrow^k$ $axSx^Ra$** or **S ⇒ bSb $\Rightarrow^k$ $bxSx^Rb$** and these are the only possibilities. But then, **S $\Rightarrow^{k+1}$ β**, **k≥0**, where **β** contains a nonterminal, iff **β** is of the form **xSx^R**, where **x∈{a,b}* and |x|=k+1**.

This proves our Lemma.

Assignment # 7.2.2 Key

2. Consider the context-free grammar **G = { {S}, {a,b}, R, S }**
   R:
   S → a | b | a a | b b | a S a | b S b
   Provide a proof that shows
   L = { w | w ∈ {a,b}⁺ and w is a palindrome }
   We wish to prove that **L(G) = L**. Specifically, we show **S ⇒* β** where **β** is over terminals, iff **β** is of the form **xx^R**, where **x∈{a,b}⁺**. By Lemma 1, **S ⇒* β**, where **β** contains a nonterminal, iff **β** is of the form **xSx^R**, where **x∈{a,b}***. **R** has four roles that can replace the one non-terminal **S** with terminals only; these are **S → a | b | a a | b b**. All strings in **L** are of one of the forms **xax^R** , **xbx^R** , **$xaax^R$** or **$xbbx^R$**, where **x∈{a,b}***. Using the Lemma and our observation about the only terminating rules in **R**, we have that **S ⇒* β** where **β** is over terminals, iff **β** is of one of the forms **xax^R** , **xbx^R** , **$xaax^R$** or **$xbbx^R$**, where **x∈{a,b}***. But this shows that the words in **L** are exactly those that can be generated in **L(G)**. Thus, we have shown that **L(G) = L** as was desired.