Assignment 1 Key

Question 2)

Prove the following: Let $\boldsymbol{R}$ be an equivalence relation over some universe $\boldsymbol{U}$, and let $\boldsymbol{C_a}$ be the class of all elements in $\boldsymbol{U}$ equivalent to the element $\boldsymbol{a}$, i.e., $\boldsymbol{C_a = \{x \mid x \in U \;\&\&\; a\, R\, x\}}$, and $\boldsymbol{C_b}$ be the class of all elements in $\boldsymbol{U}$ equivalent to the element $\boldsymbol{b}$, i.e.,
$\boldsymbol{C_b = \{x \mid x \in U \;\&\&\; b\, R\, x\}}$.
Prove that either $\boldsymbol{C_a = C_b}$ or $\boldsymbol{C_a \cap C_b = \Phi}$.

Answer

Proof: First, given any two subsets of any universe ***U***, either the subsets overlap or they have no common elements. Thus, for C_a and C_b, either $C_a \cap C_b = \Phi$, and we are done, or $C_a \cap C_b \neq \Phi$. In this latter case, there is at least one element $z \in U$, such that $z \in C_a$ and $z \in C_b$. Thus, ***a R z*** and ***b R z***. Since ***R*** is an equivalence relation, ***b R z*** implies ***z R b*** by symmetry and then ***a R b*** by transitivity and ***b R a*** by symmetry. Thus, $b \in C_a$ and $a \in C_b$, and consequently, based on symmetry, all members of C_a are in C_b, and vice versa. Mutual inclusion then means $C_a = C_b$, completing the proof. Note that reflexivity guarantees that all elements of ***U*** are in some class, so ***R*** partitions the universe ***U***.