QUICK REVIEW

THE SET OF ALGORITHMS IS <u>NOT</u> RE

NOTE! f IS AN ALG IFF $\mathrm{Dom}(f) = \mathbb{N}$

PROOF BY DIAGONALIZATION

ASSUME F AN ALG. WHOSE RANGE = SET OF ALG INDICES

DEFINE $G(x) = \Phi(F(x), x) + 1$

G IS AN ALG. SO, FOR SOME g, $F(g) = G$

BUT THEN $G(g) = \Phi(F(g), g) + 1 = G(g) + 1$ $\times\!\!\times$

REDUCTION: IF A AND B ARE TWO SETS AND $\exists$ AN ALG., f, SUCH THAT, $x \in A \Leftrightarrow f(x) \in B$ THEN $A \leq_m B$ MEANING B IS AT LEAST AS HARD AS A IN A COMPUTATIONAL SENSE (NOT NEC. A COMPLEXITY SENSE

RE-COMPLETE: IF SET S IS RE AND FOR ANY ARB. RE SET T, $T \leq_m S$, THEN S IS RE COMPLETE. (CAN DO $\leq_1$ AS WELL)

RICE'S THEOREM: IF P IS

(a) A PROPERTY THAT DIVIDES PROCEDURES INTO TWO CLASSES, $S_P = \{f \mid f \text{ HAS PROP. } P\}$; $\bar{S}_P = \{f \mid f \text{ DOES NOT}\}$

(b) P IS NON-TRIVIAL, THAT IS, $S_P \neq \emptyset$, $\bar{S}_P \neq \emptyset$

(c) IF f AND g ARE SUCH THAT $\forall x\, f(x) = g(x)$ THEN EITHER BOTH f AND g ARE IN S_P OR BOTH ARE IN $\bar{S}_P$

THEN P IS UNDECIDABLE

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RICE'S AGAIN

THREE VERSIONS OF CONDITION (c)

WEAK (DOMAIN)
IF f AND g ARE SUCH THAT $Dom(f)=Dom(g)$

WEAK (RANGE)
IF f AND g ARE SUCH THAT $RNG(f)=RNG(g)$
THEN EITHER $f,g \in S_P$ OR $f,g \in \overline{S_P}$

STRONG (MAPPING / I/O BEHAVIOR)
IF f AND g ARE SUCH THAT $\forall x\ f(x)=g(x)$
THEN EITHER $f,g \in S_P$ OR $f,g \in \overline{S_P}$

CAUTION

RICE'S CANNOT ADDRESS PROPERTIES THAT ARE ABOUT IMPLEMENTATION OR PERFORMANCE OR ABOUT SUBSETS OF PROCEDURES, E.G., CFGs OF CSGs

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RE (REC. ENUMERABLE)

S IS RE IFF
(a) $S = \emptyset$
(b) $S = \mathrm{rng}(f)$, WHERE f IS AN ALGORITHM

S IS SEMI-DECIDABLE IFF
$S = \mathrm{Dom}(g)$, WHERE g IS A PROCEDURE

S RE IFF S SEMI-DECIDABLE

KEY IS EXISTENCE OF PRIMITIVE RECURSIVE FUNCTION STP WHERE

$STP(f, \vec{x}, t)$ IFF
$\varphi_f(\vec{x})$ HALTS IN AT MOST t STEPS

A RELATED USEFUL PRF IS VALUE
$VALUE(f, \vec{x}, t) = \varphi_f(\vec{x})$ WHENEVER
$STP(f, \vec{x}, t)$; UNDEFINED OTHERWISE

$\therefore$ SEMI-DEC $\Rightarrow$ RE

ASSUME S IS SEMI-DEC. BY G_S (g_s INDEX)

IF $S=\emptyset$, THEN RE BY DEFN.

IF $S\neq\emptyset$, LET $a\in S$ BE ARBITRARY

DEFINE F_S BY

$$F_S(<x,t>) = x * STP(g_s, x, t) + a * (1 - STP(g_s, x, t))$$

F_S IS PRF THAT ENUMERATES S, PRODUCING EACH MEMBER INF. OFTEN

RE $\Rightarrow$ SEMI DEC.

IF $S=\emptyset$ THEN $S = Dom(\mu y[y=y+1])$

ASSUME $S\neq\emptyset$ AND ENUMERATED BY F_S, THEN

S IS SEMI-DECIDED BY

$$\psi_S(x) = \exists y[F_S(y)=x]$$

CAN EXTEND TO S IS SEMI-DEC (RE) IFF

S IS DOMAIN/RANGE OF A PARTIAL REC. FCN.

TAKE ψ_S ABOVE AND CHANGE TO

$$\psi_S(x) = \begin{cases} \exists y[y=y+1] & \text{IF } S=\emptyset \\ x * (\exists y[F_S(y)=x]) & \text{IF } S\neq\emptyset \end{cases}$$

RECURSIVE $\Rightarrow$ RE

S REC IFF:

$$S = \{x \in \mathbb{N} \mid f_S(x) = 1\}$$ FOR SOME ALG f_S

DEFINE

$$g_S(x) = \exists y\, [f_S(x) = 1]$$

CLEARLY

$$Dom(g_S) = S$$

S AND $\overline{S}$ RE IFF $S(\overline{S})$ RECURSIVE

LET g_S SEMI-DECIDE S; $g_{\overline{S}}$ SEMI-DECIDE $\overline{S}$

THEN S DECIDED BY

$$f_S(x) = STP(f_S, x, \mu t\, [STP(f_S, x, t) \,||\, STP(f_{\overline{S}}, x, t)])$$

$\overline{S}$ IS DECIDE BY

$$f_{\overline{S}}(x) = 1 - f_S(x)$$

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RE CHARACTERIZATION

$S \neq \emptyset$ IS RE IFF ANY OF BELOW

(a) $S = RNG(f)$ f PRF (PRIM. REC.)
(b) $S = RNG(f)$ f TOTAL RECURSIVE (ALG.)
(c) $S = RNG(f)$ f PARTIAL REC (PROCEDURE)
(d) $S = DOM(f)$ f PARTIAL REC.

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QUANTIFICATION

S IS DECIDABLE IFF $\exists$ AN ALG. χ_S (CALLED S'S CHARACTERISTIC FUNCTION) SUCH THAT

$x \in S \Leftrightarrow \chi_S(x)$ $\quad \chi_S$ IS A PREDICATE

S IS RE IFF $\exists$ AN ALG. $A_S \ni$

$x \in S \Leftrightarrow \exists t\, A_S(x,t)$

CAN SHOW WITH $A_S(x,t) = STP_{g_S}(x,t) = STP(g_S, x, t)$

WHERE $S = \mathrm{Dom}(g_S)$

$\bar{S}$ IS CO-RE IFF $\exists$ AN ALG. $A_S \ni$

$x \in \bar{S} \Leftrightarrow \forall t\, A_S(x,t)$

JUST COMPLEMENT ABOVE FOR RE

A NON-RE, NON-CO-RE PROBLEM

$f \in TOTAL \Leftrightarrow \forall x \exists t\, [STP(f,x,t)]$

CANNOT REDUCE # OF ALTERNATING QUANTIFIERS

QUANTIFICATION IS FIRST ESTIMATE OF COMP

QUANTIFICATION & UPPER BOUNDS

$\{f \mid$ IF $f(x)\downarrow$ IT DOES SO IN MORE THAN $2*x+2$ STEPS$\}$

CO-RE

$\forall x \sim STP(f, x, 2*x+2)$ CO-RE

$\{\langle f,x\rangle \mid$ IF $f(x)\downarrow$ IT DOES SO IN MORE THAN $2*x+2$ STEPS

REC

$\sim STP(f, x, 2*x+2)$

NON RE NON CO-RE

$\{f \mid \forall x\ f(x)=0\}$

$\forall x \exists t\, [STP(f,x,t) \,\&\&\, VALUE(f,x,t)=0]$

RE

$\{f \mid$ FOR SOME x $f(x)=0\}$

$\exists x \exists t\, [STP(f,x,t) \,\&\&\, VALUE(f,x,t)=0]$

LATTER TWO ARE AMENABLE TO RICE'S THEOREM. FIRST TWO ARE NOT

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USE OF QUANTIFICATION

$$HALT = \{\langle f, x\rangle \mid f(x)\downarrow\}$$

$$\langle f,x\rangle \in HALT \iff \exists t\, STP(f,x,t)$$ RE

$$TOTAL = \{f \mid \forall x\, f(x)\downarrow\}$$

$$f \in TOTAL \iff \forall x \exists t\, STP(f,x,t)$$ MIGHT BE NONRE NON-CO-RE

$$HAS ZERO = \{f \mid \text{for some } x,\ f(x)=0\}$$

$$f \in HZ \iff \exists x \exists t\, [STP(f,x,t)\ \&\&\ VALUE(f,x,t)=0]$$ AT WORST RE

$$\exists\langle x,t\rangle$$

$$DIVERGE = \{f \mid \text{for all } x\ f(x)\uparrow\}$$

$$f \in DIVERGE \iff$$

$$\forall x \forall t\, [\neg STP(f,x,t)]$$

$$\forall\langle x,t\rangle$$

AT WORST CO-RE

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CONSTANT EXECUTION TIME

$$\exists K \; \forall_{|x| \le K} x \, [STP(f, x, K)]$$

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SEMI-THUE SYSTEMS

$S = (\Sigma, R)$

R: FINITE SET OF RULES

$\alpha \to \beta \quad \alpha, \beta \in \Sigma^*$

LIKE GRAMMARS BUT NO VARIABLES

CAN SIMULATE TMs (MAYBE DO LATER)

CAN SIMULATE STs BY POST CORRESPONDENCE PROBLEM. (PCP)

$\vec{X} = (x_1, \dots, x_n)$, $\vec{Y} = (y_1, \dots, y_n)$, $n > 0$

$x_i, y_i \in \Sigma^*$

$\exists\ i_1, \dots, i_k,\ k > 0,\ 1 \le i_j \le n \ni$

$x_{i_1} \cdots x_{i_k} = y_{i_1} \cdots y_{i_k}$

$n = 3$, $\Sigma = \{a, b\}$, $\vec{X} = (aba, bb, a)$, $\vec{Y} = (bab, b, baa)$

bbaababb

SOLUTION 2, 3, 1, 2 ~~[illegible]~~a

bbaababb

IF ONE SOLN, $\exists$ AN INF. # OF SOLUTIONS

PROBLEM IS, IN GENERAL, UNDEC.

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ABIGUITY OF CFG

$S \to A \mid B$

$A \to x_i A[i] \mid x_i [i]$ } $1 \leq i \leq n$

$B \to y_i B[i] \mid y_i [i]$ }

$A \xRightarrow{*} x_{i_1} \cdots x_{i_k} [i_k] \cdots [i_1]$ } $k > 0$

$B \xRightarrow{*} y_{i_1} \cdots y_{i_k} [i_k] \cdots [i_1]$ }

AMBIGUOUS IFF SOLUTION TO INSTANCE OF PCP

INTERSECTION OF CFL

$L_A \cap L_B \neq \emptyset$ IFF SOLN OF PCP

THUS, EMPTINESS OF CSL UNDEC.

OUR EXAMPLE $\vec{X} = (aba, bb, a)$, $\vec{Y} = (bab, b, baa)$

$S \to A \mid B$

$A \to aba\,A[1] \mid bb\,A[2] \mid a\,A[3] \mid$
$aba[1] \mid bb[2] \mid a[3]$

$B \to bab\,B[1] \mid b\,B[2] \mid baa\,B[3] \mid$
$bab[1] \mid b[2] \mid baa[3]$

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NON-EMPTINESS OF CSL

$S \to x_i S y_i^R \mid x_i T y_i^R \quad 1 \le i \le n$

$aTa \to * T *$

$* a \to a *$ $\qquad \Sigma = \{*\}$

$a * \to * a$ $\qquad V = \{S, T\} \cup$ PCF ALPH.

$T \to *$

GET STRINGS $*^{2j+1}$ for SOME j's
IFF SOLN TO PCP

ABOVE ALSO SHOWS FINITENESS (INFINITENESS) OF CSL IS UNDEC.