11/8/2016

Input \ Algorithms	A_0	A_1	$\cdots$	A_k	$\cdots$
0	$A_0(0)+1$	$A_1(0)$		$A_k(0)$	
1	$A_0(1)$	$A_1(1)+1$		$A_k(1)$	
$\vdots$					
k	$A_0(k)$	$A_1(k)$		$A_k(k)+1$	
$\vdots$					

THIS IS POSSIBLE IF CAN LIST (REC. ENUM.) THE ALGORITHMS (SUBSET OF PROCEDURES, WHICH ARE ENUMERABLE).

ASSUME ALGORITHM INDICES ARE ENUMERATED BY RECURSIVE FUNCTION F

$F(x)$ = INDEX OF X-TH ALGORITHM F_x

SO $\text{UNIV}(F(x), y) = \varphi_{F_x}(y) = A_x(y)$.

DEFINE G TO DISAGREE WITH EACH $A_k(k)$

$$G(x) = \text{UNIV}(F(x), x) + 1 = \varphi_{F_x}(x) + 1 = A_x(x) + 1$$

NOW G MUST BE AN ALGORITHM AS IT'S DEFINED EVERYWHERE AND BUILT FROM OTHER ALGORITHMS. LET $A_g = G$. THAT IS,

$$G(x) = \text{UNIV}(F(g), x) = \varphi_{F_g}(x) = A_g(x)$$

BUT

$$G(g) = \text{UNIV}(F(g), g) + 1 = G(g) + 1$$

SO G CANNOT EXIST AND SO OUR ASSUMPTION IS WRONG AND THE SET OF INDICES OF ALGORITHMS IS NOT RE.

$$\text{TOTAL} = \{ f \in \mathbb{N} \mid \forall x\ \varphi_f(x)\downarrow \}$$
$$= \{ f \in \mathbb{N} \mid W_f = \mathbb{N} \}$$

IS NOT RE.

TWO USEFUL SETS

TOTAL (ABOVE) IS NON-RE

$\text{HALT} = \{ \langle f, x \rangle \mid f(x)\downarrow \}$

IS RE, NON-RECURSIVE

REDUCTION

HALT $\leq$ TOTAL

LET $\langle f, x \rangle$ BE ARB. PAIR OF NAT NUMBERS

DEFINE $f_x(y) = f(x) \quad \forall y \in \mathbb{N}$

$\langle f, x \rangle \in$ HALT IFF $f(x)\downarrow$

IFF $\forall y\ f_x(y)\downarrow$

IFF $f_x \in$ TOTAL

REALLY φ_f AND φ_{f_x}

ANY ALGORITHM THAT SOLVES TOTAL
CAN BE USED TO SOLVE HALT,
SO HALT $\leq$ TOTAL
AND SINCE HALT IS UNDEC., SO IS TOTAL

CANNOT SHOW TOTAL $\leq$ HALT
AS TOTAL IS NOT EVEN RE,
BUT HALT IS

11/8/2016

MoreReductions

HALT $\leq$ ZERO $= \{f \mid \forall x\ \varphi_f(x) = 0\}$

LET $\langle f, x \rangle$ BE ARB. PAIR

DEFINE $\forall y\ f_x(y) = f(x) - f(x)$

$\langle f, x \rangle \in$ HALT IFF $f(x)\downarrow$ IFF $f(x) - f(x) = 0$

IFF $\forall y f_x(y) = 0$ IFF $f_x \in$ ZERO

So ZERO IS UNDECIBLE, BUT IT'S WORSE

TOTAL $\leq$ ZERO

LET f BE ARBITRARY INDEX ($f \in \mathbb{N}$)

DEFINE $\forall x\ g(x) = f(x) - f(x)$

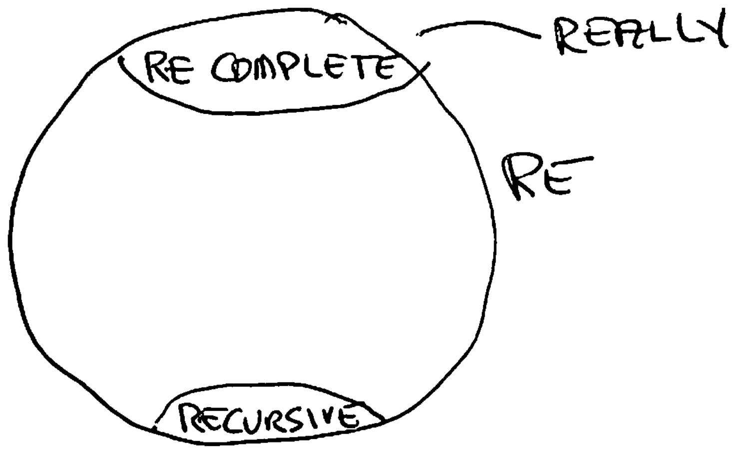
$f \in$ TOTAL IFF $\forall x f(x)\downarrow$

IFF $\forall x\ f(x) - f(x) = 0$

IFF $\forall x\ g(x) = 0$

IFF $g \in$ ZERO

RE-COMPLETE SETS



S IS RE-COMPLETE IFF

(a) S IS RE

(b) FOR ANY RE SET T, $T \leq S$

WE FOCUS ON $\leq_M$ OR EVEN $\leq_1$ FOR NOW

HALT $= K_0 = \{\langle f, x\rangle \mid \varphi_f(x)\downarrow\}$ IS RE-COMPLETE

(a) IT IS RE (CAN SEMI-DECIDE)

(b) LET T BE AN ARB RE SET
BY DEFINITION $\exists$ AN EFF PROC φ_t SUCH THAT $\mathrm{Dom}(\varphi_t) = T$, OR EQUIV. $\exists$ AN INDEX $t \ni T = W_t$ (ENUMERATION TH)

$x \in T$ IFF $x \in \mathrm{Dom}(\varphi_t)$
IFF $\varphi_t(x)\downarrow$
IFF $\langle t, x\rangle \in K_0 =$ HALT

SO $T \leq_1 K_0$

SINCE T IS ARB. RE, THIS SHOW K_0 IS RE (1-1, m-1, TURING) COMPLETE.

11/8/2016

$K = \{f \mid \varphi_f(f)\downarrow\}$ IS RE COMPLETE

JUST SHOW $K_0 \le K$

LET $\langle f,x\rangle$ BE ARB. PAIR FROM $\mathbb{N}\times\mathbb{N}$

DEFINE $\forall y\ f_x(y) = \varphi_f(x)$

LET INDEX OF f_x BE f_x (OVERLOAD)

$\langle f,x\rangle \in K_0$ IFF $x \in Dom(\varphi_f)$ IFF $\forall y\,[\varphi_{f_x}(y)\downarrow] \Rightarrow f_x \in K$

$\langle f,x\rangle \notin K_0$ IFF $x \notin Dom(\varphi_f)$ IFF $\forall y\,[\varphi_{f_x}(y)\uparrow] \Rightarrow f_x \notin K$

$K_0 \le_1 K$, $K_0 \le_m K$, $K_0 \le_{TURING} K$

SO K_0 IS RE (1-1, M-1, TURING) COMPLETE

11/8/2016

RICE'S THEOREM

ONLY APPLIES TO PROPERTIES OF PROCEDURES.

ONLY APPLIES IF WE CANNOT DIFFERENTIATE PROCEDURES WITH SAME I/O BEHAVIORS

ONLY APPLIES IF PROPERTY IS NON-TRIVIAL (SOME PROCEDURES HAVE IT; SOME DON'T)

NOTE 1: IFF NON-TRIVIAL THEN $\exists$ SOME INDEX $r \ni$
$r \in S_P = \{ f \mid \varphi_f$ HAS PROPERTY $P \}$

NOTE 2: ALL PROCEDURE INDICES IN
EMPTY $= \{ f \mid \forall x \ \varphi_f(x)\uparrow \}$
ARE EITHER IN S_P OR $\overline{S_P}$
WLOG, ASSUME THEY ARE IN $\overline{S_P}$

LET x, y BE ARB AND DEFINE
$$F_{r,x,y}(z) = \varphi_x(y) - \varphi_x(y) + \varphi_r(z)$$

11/8/2016

ANALYZE $F_{r,x,y}(z) = \varphi_x(y) - \varphi_x(y) + \varphi_r(z)$

1. IF $\langle x,y \rangle \in$ HALT THEN

   $\forall z\ F_{r,x,y}(z) = \varphi_r(z)$

2. IF $\langle x,y \rangle \notin$ HALT THEN

   $\forall z\ F_{r,x,y}(z)\uparrow$

IN CASE 1, $F_{r,x,y} \in S_P$

IN CASE 2, $F_{r,x,y} \notin S_P$

THUS,

$\text{HALT} \leq S_P$