

Assignment # 2.1 Key

Prove or disprove, if p and q are distinct prime numbers ($p \neq q$), then $\sqrt[pq]{pq}$ is irrational.

Proof:

Assume $\sqrt[pq]{pq}$ is a rational number. Let a/b be the reduced fraction (no common prime factors) that equals $\sqrt[pq]{pq}$.

$$\sqrt[pq]{pq} = a/b$$

: assumption (note $a \neq b$, as then $pq=1$)

$$pq = a^2/b^2$$

: square both sides

$$pq b^2 = a^2$$

: multiply both sides by b^2

$$pq b^2 = (kpq)^2$$

: for some k , as p and q must be prime factors of a since a and b have no common prime factors

$$b^2 = k^2 pq$$

: divide both sides by pq

But then p and q must be prime factors of both a and b . But then a/b is not reduced as both have common prime factors p and q . This contradicts our original assumption that $\sqrt[pq]{pq}$ is rational, so it is irrational.

Assignment # 2.2 Key

Prove, if L is a language over Σ then

$\lim(n \rightarrow \infty) L^n = \Sigma^*$ iff $(\Sigma \cup \{\lambda\}) \subseteq L$.

Note: $L^k = \{ x_1 x_2 \dots x_k \mid x_1, x_2, \dots, x_k \in L \}$

Proof:

Assume $(\Sigma \cup \{\lambda\}) \subseteq L$. As $\lambda \in L$, $L^k \subseteq L^{k+1}$, for all $k \geq 0$, since $L^k \cdot \lambda = L^k$.

By definition $\Sigma^* = \{\lambda\} \cup \Sigma \cup \Sigma^2 \cup \dots \cup \Sigma^k \cup \dots = \bigcup_{k=0}^{\infty} \Sigma^k$

Since $(\Sigma \cup \{\lambda\}) \subseteq L$ we have that, for each $k \geq 0$, $L^k \subseteq L^{k+1} \subseteq \lim(n \rightarrow \infty) L^n$.

Thus, for each $k \geq 0$, $\Sigma^k \subseteq L^k \subseteq \lim(n \rightarrow \infty) L^n$ and so $\Sigma^* = \bigcup_{k=0}^{\infty} \Sigma^k \subseteq \lim(n \rightarrow \infty) L^n$.

Given that L is a language over Σ then $L \subseteq \Sigma^*$, by definition, and therefore

$L^n \subseteq \Sigma^*$ for all $n \geq 0$ and so $\lim(n \rightarrow \infty) L^n \subseteq \lim(n \rightarrow \infty) \Sigma^* \subseteq \Sigma^*$.

Putting this together, we have $\lim(n \rightarrow \infty) L^n = \Sigma^*$.

Thus, **$(\Sigma \cup \{\lambda\}) \subseteq L$ implies $\lim(n \rightarrow \infty) L^n = \Sigma^*$.**

Assume $\lim(n \rightarrow \infty) L^n = \Sigma^*$. Clearly L^{k+1} cannot contain strings shorter than those found in L^k as $L^{k+1} = L^k L$. Thus if any of $(\Sigma \cup \{\lambda\})$ is missing from L , then that element is also missing from all L^k , $k > 1$ and so $\lim(n \rightarrow \infty) L^n \neq \Sigma^*$, which is a contradiction. **Thus, $\lim(n \rightarrow \infty) L^n = \Sigma^*$ implies $(\Sigma \cup \{\lambda\}) \subseteq L$.**

Second Solution for 2.2

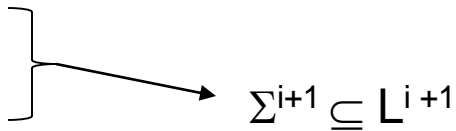
- Proof by Induction :

- Assume $(\Sigma \cup \{\lambda\}) \subseteq L$.

We want to prove that, $\forall i \geq 0, \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \dots \cup \Sigma^i \subseteq L^i$:

- Base: $i = 0$: as $\Sigma^0 = \{\lambda\} \subseteq L^0$ by definition of 0-th power (even $\{\}^0$ contains λ)
- Induction Hypothesis: Assume for some $i \geq 0, \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \dots \cup \Sigma^i \subseteq L^i$
- Induction Step: Show for $i+1$

- $\Sigma^i \subseteq L^i$ by IH
- $\Sigma \subseteq L$ assumed
- Because $\lambda \in L, \forall k \geq 0, L^k \subseteq L^{k+1}$
- Hence, $\Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \dots \cup \Sigma^i \cup \Sigma^{i+1} \subseteq L^i \cup \Sigma^{i+1} \subseteq L^{i+1}$



As i goes to infinity The left side becomes Σ^* and right side becomes $\lim_{n \rightarrow \infty} L^n$. Therefore $\Sigma^* \subseteq \lim_{n \rightarrow \infty} L^n$.

$\lim_{n \rightarrow \infty} L^n \subseteq \Sigma^*$ is trivial. Now we can conclude that $\Sigma^* = \lim_{n \rightarrow \infty} L^n$.