# Assignment # 2.1 Key

Prove or disprove, if p and q are distinct prime numbers ($p \neq q$), then $\sqrt{(pq)}$ is irrational.

Proof:

Assume $\sqrt{pq}$ is a rational number. Let a/b be the reduced fraction (no common prime factors) that equals $\sqrt{(pq)}$.

| | |
|---|---|
| $\sqrt{(pq)} = a/b$ | : assumption (note $a \neq b$, as then $pq=1$) |
| $pq = a^2/b^2$ | : square both sides |
| $pqb^2 = a^2$ | : multiply both sides by $b^2$ |
| $pqb^2 = (kpq)^2$ | : for some k, as p and q must be prime factors of a since a and b have no common prime factors |
| $b^2 = k^2pq$ | : divide both sides by pq |

But then p and q must be prime factors of both a and b. But then a/b is not reduced as both have common prime factors p and q. This contradicts our original assumption that $\sqrt{(pq)}$ is rational, so it is irrational.

# Assignment # 2.2 Key

Prove, if L is a language over $\Sigma$ then
$\lim(n\to\infty)\ L^n = \Sigma^*$ iff $(\Sigma\cup\{\lambda\}) \subseteq L$.
Note: $L^k = \{ x_1x_2\ldots x_k \mid x_1,x_2,\ldots,x_k \in L \}$

Proof:

Assume $(\Sigma\cup\{\lambda\}) \subseteq L$. As $\lambda \in L$, $L^k \subseteq L^{k+1}$, for all $k\geq 0$, since $L^k \cdot \lambda = L^k$.
By definition $\Sigma^* = \{\lambda\}\cup\Sigma\cup\Sigma^2\cup \ldots \cup\Sigma^j\cup \ldots = \bigcup_{k=0}^{\infty} \Sigma^k$
Since $(\Sigma\cup\{\lambda\}) \subseteq L$ we have that, for each $k\geq 0$, $L^k \subseteq L^{k+1} \subseteq \lim(n\to\infty)\ L^n$.
Thus, for each $k\geq 0$, $\Sigma^k\subseteq L^k\subseteq\lim(n\to\infty)\ L^n$ and so $\Sigma^*=\bigcup_{k=0}^{\infty} \Sigma^k \subseteq \lim(n\to\infty)\ L^n$.
Given that L is a language over $\Sigma$ then $L\subseteq\Sigma^*$, by definition, and therefore $L^n \subseteq \Sigma^*$ for all $n \geq 0$ and so $\lim(n\to\infty)\ L^n \subseteq \lim(n\to\infty)\ \Sigma^* \subseteq \Sigma^*$.
Putting this together, we have $\lim(n\to\infty)\ L^n = \Sigma^*$.
Thus, $(\Sigma\cup\{\lambda\}) \subseteq L$ implies $\lim(n\to\infty)\ L^n = \Sigma^*$.

Assume $\lim(n\to\infty)\ L^n = \Sigma^*$. Clearly $L^{k+1}$ cannot contain strings shorter than those found in $L^k$ as $L^{k+1} = L^k L$. Thus if any of $(\Sigma\cup\{\lambda\})$ is missing from L, then that element is also missing from all $L^k$, $k > 1$ and so $\lim(n\to\infty)\ L^n \neq \Sigma^*$, which is a contradiction. Thus, $\lim(n\to\infty)\ L^n = \Sigma^*$ implies $(\Sigma\cup\{\lambda\}) \subseteq L$.

# Second Solution for 2.2

- Proof by Induction :
- Assume $(\Sigma \cup \{\lambda\}) \subseteq L$.
  We want to prove that, $\forall i \geq 0,\ \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots \cup \Sigma^i \subseteq L^i$ :
- Base: I = 0: as $\Sigma^0 = \{\lambda\} \subseteq L^0$ by definition of 0-th power (even $\{\}^0$ contains $\lambda$)
- Induction Hypothesis: Assume for some $i \geq 0$, $\Sigma^0 \cup \Sigma \cup \Sigma^2 \cup \ldots \cup \Sigma^i \subseteq L^i$
- Induction Step: Show for i+1
  - $\Sigma^i \subseteq L^i$ by IH
  - $\Sigma \subseteq L$ assumed

    → $\Sigma^{i+1} \subseteq L^{i+1}$
  - Because $\lambda \in L$, $\forall k \geq 0$, $L^k \subseteq L^{k+1}$
  - Hence, $\Sigma^0 \cup \Sigma \cup \Sigma^2 \cup \ldots \cup \Sigma^i \cup \Sigma^{i+1} \subseteq L^i \cup \Sigma^{i+1} \subseteq L^{i+1}$

  As i goes to infinity The left side becomes $\Sigma^*$ and right side becomes $\lim(n \to \infty)\ L^n$ . Therefore $\Sigma^* \subseteq \lim(n \to \infty)\ L^n$ .

  $\lim(n \to \infty)\ L^n \subseteq \Sigma^*$ is trivial. Now we can conclude that $\Sigma^* = \lim(n \to \infty)\ L^n$.