Sample Assignment # 1.1

1. Prove that, for sets A and B,
   $A=B$ if and only if $(A\cap\sim B)\cup(\sim A\cap B)=\emptyset$,
   where ~S is the complement of S

Part 1) Prove if $A = B$, then $(A\cap\sim B)\cup(\sim A\cap B)=\emptyset$

Assume $A=B$ then $(A\cap\sim B)\cup(\sim A\cap B)= (A\cap\sim A)\cup(\sim A\cap A)$

Now, any set intersected with its complement must be empty by the definition of complement, so $(A\cap\sim A)=\emptyset$ and $(\sim A\cap A)=\emptyset$ and thus their union is also empty, proving that $A = B$ implies $(A\cap\sim B)\cup(\sim A\cap B)=\emptyset$.

Part 2) Prove if $(A\cap\sim B)\cup(\sim A\cap B)=\emptyset$, then $A = B$

Assume $(A\cap\sim B)\cup(\sim A\cap B)=\emptyset$ then, by definition of union, $(A\cap\sim B)=\emptyset$ and $(\sim A\cap B)=\emptyset$ else the union would have at least one element in it. This in turn implies that no element of A is in the complement of B and no element of B is in the complement of A. Thus, all elements of A are in B and all elements of N are in A. Stated more formally, $A \subseteq B$ and $B \subseteq A$. But, mutual containment is the definition of set equality and so $A = B$. proving that $(A\cap\sim B)\cup(\sim A\cap B)=\emptyset$ implies $A = B$.

Sample Assignment # 1.2

2. Prove, If S is any finite set with $|S| = n$, then $|S\times S\times S\times S| \leq |\mathcal{P}(S)|$, for all $n \geq N$, where N is some constant, the minimum value of which you must discover and use as the basis for your proof.

Proof:

(This is the same as showing $n^4 \leq 2^n$, for all $n \geq N$. We shall show this is true when $N=16$.)

Basis: $16^4 = (2^4)^4 = 2^{16} \leq 2^{16}$. This proves the base case. Note: that

$15^4 = 50625$ and $2^{15} = 32768$ and so $N=15$ fails.

I.H. Assume for some K, $K \geq 16$, $K^4 \leq 2^K$.

I.S.(K+1) : $(K+1)^4 = K^4 + 4K^3 + 6K^2 + 4K + 1$

$\leq K^4 + 4K^3 + 6K^3 + 4K^3 + K^3$ since $K \geq 1$

$= K^4 + 15K^3 \leq K^4 + K^4$ since $K \geq 16$

$\leq 2^K + 2^K$ by IH

$= 2^{K+1}$

Thus, $(K+1)^4 \leq 2^{K+1}$ and the I.S. is proven.

Sample Assignment # 1.3

3. Consider the function *pair*: $\boldsymbol{N} \times \boldsymbol{N} \rightarrow \boldsymbol{N}$
defined by *pair*(x,y) = 2^x (2y + 1) – 1
Show that *pair* is a surjection (onto ***N***).
Note: It's actually a bijection (1-1 onto ***N***), but I am not asking you to show that.

Proof:

Case 1: All even numbers are in range.

Let x=0. Then 2^x (2y + 1) – 1 = 2y + 1 – 1 = 2y where y ≥ 0

Since y ranges over the natural numbers, 2y ranges over all even numbers and case 1 is shown.

Case 2: All odd numbers are in range.

Let x>0. Odd numbers are all those of the form 2z-1, z>0. That is, they have a non-trivial even factor and an odd factor that could be just 1. Essentially, then, every odd number is 1 less than some non-zero even number. But, every non-zero even number has a factorization that is of the form 2^x (2y + 1), where x>0 and y ≥ 0. This shows that 2^x (2y + 1) – 1 ranges over all odd numbers, when x>0 and case 2 is shown.

Sample Assignment # 1.3

3. Consider the function *pair*: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$
   defined by *pair*(x,y) = $2^x (2y + 1) – 1$
   Show that *pair* is a surjection (onto $\mathbb{N}$).
   Note: It's actually a bijection (1-1 onto ***N***), but I am not asking you to show that.

Simpler Proof Informed by case 2 on previous page:

All natural numbers are in range of $2^x (2y + 1) – 1$.

We show this by proving that all positive natural numbers are in the range of $2^x (2y + 1)$. We note that every non-zero natural number has a unique factorization that is of the form $2^x (2y + 1)$, where the first term captures the number's even part (or x=0 if not even) and the second part captures its odd part. This shows that $2^x (2y + 1) – 1$ ranges over all natural numbers and so the range of *pair* is the set of all natural numbers. Note, we actually show uniqueness here based on the unique prime factorization theorem and so *pair* is not just a surjection; it is a bijection.