Practice Assignment # 8

1. **Use Rice's Theorem to show that { f | ∃x f(x) = 0 } is undecidable**
2. **Use Rice's Theorem to show that {f | ∀x f(x+1)=f(x)+1} is undecidable**
3. **Use quantification of an algorithmic predicate to estimate the complexity (decidable, re, co-re, non-re) of each of the following, (a)-(d):**
   a) **{ f | for all input x, f(x) = f(0), that is f is a constant function }**
   b) **{ f | for two unique input values, x,y, f(x) = f(y) }**
   c) **{ <f,x> | f(x) takes at least 10 time steps before converging }**
   d) **{ <f,x> | f(x)↑ }**
4. **Let sets A and B each be re non-recursive (undecidable).**
   Consider C = A ∩ B. For (a)-(c), either show sets A and B with the specified property or demonstrate that this property cannot hold.
   a) **Can C be recursive?**
   b) **Can C be re non-recursive (undecidable)?**
   c) **Can C be non-re?**

Assignment # 8.1

1. Use Rice's Theorem to show that $\{ f \mid \exists x [\varphi_f(x) = 0] \}$ is undecidable

Call this set SI = $\{ f \mid \exists x [\varphi_f(x) = 0] \}$.
Let f be an arbitrary index (natural number). f is in SI iff $\exists x [\varphi_f(x) = 0]$
First, SI is non-trivial as
$Z(x) = 0$ is in SI and $I(x) = x$ is not in SI
Second, SI is an I/O property as
Let f,g be arbitrary indices (natural numbers) such that $\forall x\ \varphi_f(x) = \varphi_g(x)$.
φ_f is in SI iff $\exists x\ \varphi_f(x)=0$. Let one of the x's with this property be x_0.
That is, $\varphi_f(x_0) = 0$.
Since $\forall x\ \varphi_f(x) = \varphi_g(x)$, $\varphi_g(x_0) = 0$.
But then, φ_f is in SI implies φ_g is in SI.
If, on the other hand, $\sim\exists x\ \varphi_f(x) = 0$, then $\sim\exists x\ \varphi_g(x) = 0$,
and so if $f \notin$ SI then $g \notin$ SI .
Combining these $f \in$ SI iff $g \in$ SI
The above shows that SI satisfies both conditions for Rice's Theorem, and hence SI is undecidable.

Assignment # 8.2

2. Use Rice's Theorem to show that { f | $\forall$x [φ_f(x+1) = φ_f(x) + 1] } is undecidable

Call this set MI = { f | $\forall$x [φ_f(x+1) = φ_f(x) + 1] }.

Let f be an arbitrary index (natural number). f is in MI iff $\forall$x φ_f(x+1) = φ_f(x) +1

First, MI is non-trivial as

I(x) = x is in MI and Z(x) = 0 is not in MI

Second, MI is an I/O property as

Let f,g be arbitrary indices (natural numbers) such that $\forall$x φ_f(x) = φ_g(x).

φ_f is in MI iff $\forall$x φ_f(x+1)=φ_f(x)+1 iff $\forall$x φ_g(x+1)=φ_g(x)+1 , since $\forall$x φ_f(x) = φ_g(x)

But then, φ_f is in MI iff φ_g is in MI.

The above shows that MI satisfies both conditions for Rice's Theorem, and hence MI is undecidable.

Assignment # 8.3

3. **Use quantification of a algorithmic predicate to estimate the complexity (decidable, re, co-re, non-re) of each of the following, (a)-(d):**
   a) **{ f | for all input x, f(x) = f(0), that is f is a constant function }**
      { f | ∀x∃t [STP(f,x,t) && STP(f,0,t) && (VALUE(f,x,t) == VALUE(f,0,t))] }
      Hence non-re
   b) **{ f | for two unique input values, x,y, f(x) = f(y) }**
      { f | ∃<x,y,t> [STP(f,x,t) && STP(f,y,t) && (VALUE(f,x,t) == VALUE(f,y,t))] }
      re
   c) **{ <f,x> | f(x) takes at least 10 time steps before converging }**
      { <f,x> | ~STP(f,x,9) }
      decidable
   d) **{ <f,x> | f(x)↑ }**
      { <f,x> | ∀t [~STP(f,x,t)] }
      co-re

Assignment # 8.4

4. **Let sets A and B each be re non-recursive (undecidable).**
   Consider $C = A \cap B$. For (a)-(c), either show sets A and B with the specified property or demonstrate that this property cannot hold.
   a) **Can C be recursive?**
      Yes. Let $A = \{ 2x \mid x \in \text{HALT} \}$; $B = \{ 2x+1 \mid x \in \text{HALT} \}$. Both A and B are many-one equivalent to Halt and so both are re non-recursive, but $A \cap B = \varnothing$, which is recursive (decidable).
   b) **Can C be re non-recursive (undecidable)?**
      Yes. Let A = B = HALT. Both A and B are re non-recursive, and $A \cap B$ = HALT, which is re non- recursive (undecidable).
   c) **Can C be non-re?**
      No. The re sets are closed under intersection by the following argument. Let A and B be arb. Re sets, Let these be the domains of two procedures g_A and g_B, respectively. Define $g_{A \cap B}(x) = g_A(x) * g_A(x)$. Clearly the domain of $g_{A \cap B}$ is the intersection of the domains of g_A and g_B and so is $A \cap B$, showing this set is re.