Assignment # 7

Known Results:

Halt = { f,x | f(x)↓ } is re (semi-decidable) but undecidable

Total = { f | ∀x f(x)↓ } is non-re (not even semi-decidable)

1. **Use reduction from Halt to show that one cannot decide { f | ∃x f(x) = x } is undecidable**
2. **Show that { f | ∃x f(x) = x } reduces to Halt. (1 plus 2 show they are equally hard)**
3. **Use reduction from Halt to show that one cannot decide { f | ∀x f(x+1)=2*f(x)+1} Note that f(0) can be any value.**
4. **Use Reduction from Total to show that { f | ∀x f(x+1)=2*f(x)+1} is not even re**
5. **Show { f | ∀x f(x+1)=2*f(x)+1} reduces to Total. (4 plus 5 show they are equally hard)**

Assignment # 7.1

1. Use reduction from Halt to show that one cannot decide { f | ∃x f(x) = x } is undecidable

Let f,x be an arbitrary pair of natural numbers. <f,x> is in Halt iff $\varphi_f(x)\downarrow$

Define g by $\varphi_g(y) = \varphi_f(x) - \varphi_f(x) + y$, for all y.

Clearly, $\varphi_g(y) = y$, for all y, iff $\varphi_f(x)\downarrow$ and $\varphi_g(y)\uparrow$, for all y, otherwise.

Summarizing, <f,x> is in Halt iff g is in { f | ∃x $\varphi_f(x) = x$ } and so

Halt $\leq_m$ { f | ∃x $\varphi_f(x) = x$ } as we were to show.

Note: I have not overloaded the index of a function with the function in my proof, but I do not mind if you do such overloading.

Assignment # 7.2

2. Show that { f | ∃x f(x) = x } reduces to Halt. (1 plus 2 show they are equally hard)

Let f be an arbitrary natural number. f is in { f | ∃x φ_f(x) = x } iff ∃x φ_f(x)=x.

Define g by φ_g(y) = μ<z,t> [STP(f,z,t) & (VALUE(f,z,t) = z)], for all y.

Here μ<z,t>[P] means the least <z,t> having this property, P. The pairing allows us to search all possible pairs z,t until we find one that verifies ∃z φ_f(z) = z . If no such pair exists, this runs forever.

Clearly, φ_g(y)↓, for all y, iff, some some x, φ_f(x)=x and φ_g(y)↑, for all y, otherwise.

Thus, φ_g(0)↓ iff ∃x φ_f(x) = x

Summarizing, f is in { f | ∃x φ_f(x) = x } iff <g,0> is in Halt, and so

{ f | ∃x φ_f(x) = x } $\leq_m$ Halt as we were to show.

Assignment # 7.3

3. Use reduction from Halt to show that one cannot decide $\{ f \mid \forall x\ f(x+1)=2*f(x)+1\}$ Note that f(0) can be any value

Let f,x be an arbitrary pair of natural numbers. <f,x> is in Halt iff $\varphi_f(x)\downarrow$

Define g by $\varphi_g(y) = \varphi_f(x) - \varphi_f(x) + 2^y - 1$, for all y.

Clearly, $\varphi_g(y) = 2^y - 1$, for all y, iff $\varphi_f(x)\downarrow$ and $\varphi_g(y)\uparrow$, for all y, otherwise.

Thus, $\varphi_g(0)= 0$ and $\forall x\ \varphi_g(x+1)=2^{(x+1)}-1 = 2*2^x-1 = 2*(2^x +1) + 1 = 2*\varphi_g(x)+1$, and hence has the property that $\forall x\ \varphi_g(x+1)=2*\varphi_g(x)+1$ when <f,x> is in Halt or is undefined everywhere when <f.x> is not in Halt.

Summarizing, <f,x> is in Halt iff g is in $\{ f \mid \forall x\ \varphi_f (x+1) = 2*\varphi_f(x) + 1 \}$ and so Halt $\leq_m \{ f \mid \forall x\ \varphi_f(x+1)=\varphi_f(x)+1 \}$ as we were to show.

Assignment # 7.4

4. Use Reduction from Total to show that { f | ∀x f(x+1)=2*f(x)+1 } is not even re

Let f be an arbitrary natural number. f is in Total iff ∀x $\varphi_f(x)\downarrow$

Define g by $\varphi_g(x) = \varphi_f(x) - \varphi_f(x) + 2$^x - 1.

Clearly from analysis in 7.3, $\varphi_g(x) = 2*\varphi_g(x)+1$ iff $\varphi_f(x)\downarrow$ and $\varphi_g(x)\uparrow$ iff $\varphi_f(x)\uparrow$.

Thus, φ_g has property that ∀x $\varphi_g(x+1) = 2*\varphi_g(x) + 1$, when f is in Total, and is undefined everywhere, otherwise.

Summarizing, f is in Total iff g is in { f | ∀x $\varphi_f(x+1) = 2*\varphi_f(x) + 1$} and so

Total $\leq_m$ { f | ∀x $\varphi_f(x+1) = 2*\varphi_f(x) + 1$ } as we were to show.

Assignment # 7.5

5. Show { f | ∀x f(x+1)=2*f(x)+1 } reduces to Total. (4 plus 5 show they are equally hard)

Let f be an arbitrary natural number. f is in our target set iff
∀x φ_f (x+1)=2*φ_f(x)+1

Define g by $\varphi_g(0) = \varphi_f(0)$; $\varphi_g(x+1) = \mu y\ [\varphi_f(x+1) = 2*\varphi_f(x) + 1] + \varphi_f(x+1)$.

Clearly, $\varphi_g(0) = \varphi_f(0)$ and $\varphi_g(x+1) = \varphi_f(x+1)$ iff $\forall y \leq x\ [\varphi_f(y+1) = 2*\varphi_f(y) + 1\ ]$.

This means that φ_g cannot be in Total if φ_f diverges anywhere or if φ_f is not increasing by 2 times the previous value plus 1. Otherwise φ_g is another implementation of φ_f

Thus, φ_g is in Total, when f is in Total and is monotonically increasing by 2 times the previous value plus 1. Otherwise, φ_g is undefined starting at the first value where φ_f either diverges or fails to be double plus 1 for each increment of 1 in its input.

Summarizing, f is in { f| ∀x φ_f (x+1) = 2*φ_f(x) + 1 } iff g is in Total and so

{ f | ∀x φ_f (x+1) = 2*φ_f(x) + 1 } $\leq_m$ Total as we were to show.