Assignment # 1.1 Key

1. Prove or disprove that, for sets A and B,
   $A=B$ if and only if $(A \cap \sim B) \cup (A \cap B) = A$

Part 1) Prove if $A = B$, then $(A\cap\sim B)\cup(A\cap B)=A$

Assume $A=B$ then $(A\cap\sim B)\cup(A\cap B)= (A\cap\sim A)\cup(A\cap A)$

Now, any set intersected with its complement must be empty by the definition of complement, so $(A\cap\sim A)=\emptyset$ and $(A\cap A)=A$ and thus their union is also A, proving that $A = B$ implies $(A\cap\sim B)\cup(A\cap B)=A$.

Part 2) Disprove if $(A\cap\sim B)\cup(A\cap B)=A$, then $A = B$

Assume $(A\cap\sim B)\cup(A\cap B)=A$ and choose $B = \varnothing = \{\}$. Then $\sim B$ = Everything
Now, $A\cap\sim B = A$, since the intersection of any set, A, with the entire universe of discourse is just A.
Also, $A\cap B = \varnothing$, since the intersection of any set, A, with $\varnothing$ is $\varnothing$.

Now choose A to be any non-empty set and $(A\cap\sim B)\cup(A\cap B)=A$ whenever $B = \varnothing$. But then $A \neq B$, and the implication does not hold.

Bottom line is that this hypothesis is false.

Assignment # 1.2 Key

2. Prove that, for Boolean (T/F) variables P and Q,
   $((P \Rightarrow Q) \Rightarrow Q) \Leftrightarrow (P \vee Q)$
   $\vee$ is logical or; $\Rightarrow$ is logical implication; $\Leftrightarrow$ is logical equivalence.

Proof:

a) Let P be true then $(P \Rightarrow Q) = Q$ since, if Q is true we have $T \Rightarrow T = T$ and if Q is false we have $T \Rightarrow F = F$. Thus, when P = true, $((P \Rightarrow Q) \Rightarrow Q) = Q \Rightarrow Q = T$.
Moreover, if P is true then so is $P \vee Q$, so we have $T \Leftrightarrow T = T$.

b) Let Q be true then $(P \Rightarrow Q) = T$ since, anything can imply true. Thus, when P = true, $((P \Rightarrow Q) \Rightarrow Q) = T \Rightarrow Q = T$.
Moreover, if Q is true then so is $P \vee Q$, so we have $T \Leftrightarrow T = T$.

c) The only remaining case is when P and Q are both false.
If this is so then $(P \Rightarrow Q) \Rightarrow Q = (F \Rightarrow F) \Rightarrow F = T \Rightarrow F = F$
and $P \vee Q = F \vee F = F$, so we have $F \Leftrightarrow F = T$.

This covers all cases and so $((P \Rightarrow Q) \Rightarrow Q) \Leftrightarrow (P \vee Q)$ is a tautology.

Assignment # 1.3 Key

3. Prove, If S is any finite set with $|S| = n$, then $|S\times S\times S\times S \times S| \leq |P(S)|$, for all $n \geq N$, where N is some constant, the minimum value of which you must discover and use as the basis for your proof.

Proof:

(This is the same as showing $n^5 \leq 2^n$, for all $n \geq N$. We shall show this is true when N=23.)

Basis: $23^5 = 6{,}436{,}343 \leq 8{,}388{,}608 = 2^{23}$. This proves the base case. Note: that $22^5 = 5{,}153{,}632$ and $2^{22} = 4{,}194{,}304$ and so N=22 fails.

I.H. Assume for some K, $K \geq 23$, $K^5 \leq 2^K$.

I.S.(K+1) : $(K+1)^5 = K^5 + 5K^4 + 10K^3 + 5K^2 + 1$

$\leq K^5 + 5K^4 + 10K^4 + 5K^4 + K^4$ since $K \geq 1$

$= K^5 + 21K^4 \leq K^5 + K^5$ since $K \geq 23$

$\leq 2^K + 2^K$ by IH

$= 2^{K+1}$

Thus, $(K+1)^5 \leq 2^{K+1}$ and the I.S. is proven.

Assignment # 1.4 Key

4. Consider the function *pair*: $\mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$
   defined by *pair*(x,y) = $2^x (2y + 1) – 1$
   Show that *pair* is a bijection (1-1 onto ***N***).
   Note: I already showed this is a surjection in the Sample, so your assignment is to show it is an injection (1-1), not just onto.

Proof:

Let (x,y) and (x',y') be two pairs of natural numbers such that *pair*(x,y) = *pair*(x,y'). This means that $2^x (2y + 1) – 1 = 2^{x'} (2y' + 1) – 1$ or equivalently that $2^x (2y + 1) = 2^{x'} (2y' + 1)$, but unique prime factorization says that all non-zero natural numbers can be uniquely factored as the product of primes. Said differently, each non-zero natural number can be factored into its even components (a unique power of 2) and its odd components (a product of unique odd primes).

Thus, $2^x (2y + 1) = 2^{x'} (2y' + 1)$ implies x = x' and y = y'. This shows that *pair* is an injection (1-1), as desired.