COT 3100 Recitation Solutions: Induction

1) Use induction on n to prove the following summation for all positive integers n:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

<u>Solution</u>

Base case: n = 1 LHS = $\sum_{i=1}^{1} i^3 = 1^3 = 1$, RHS = $\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1.$

The given statement is true for n = 1.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

Inductive Step: Prove for n = k+1 that

$$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$$

$\sum_{i=1}^{k+1} i^3 = \left(\sum_{i=1}^{k} i^3\right) + (k+1)^3$

$= \frac{k^2(k+1)^2}{4} + (k+1)^3$, using the inductive hypothesis

$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$, getting a common denominator

$= \frac{(k+1)^2}{4}(k^2 + 4(k+1))$, factoring out $\frac{(k+1)^2}{4}$

$= \frac{(k+1)^2}{4}(k^2 + 4k + 4)$, multiply out

$= \frac{(k+1)^2(k+2)^2}{4}$, factoring

2) The n^{th} Harmonic number, denoted H_n is defined as follows: $H_n = \sum_{i=1}^{n} \frac{1}{i}$

Prove that the following equation is true for all positive integers n, using induction on n:

$$\sum_{i=1}^{n} \frac{i}{i+1} = (n+1) - H_{n+1}$$

Solution

Base case: n=1 LHS = $\sum_{i=1}^{1} \frac{i}{i+1} = \frac{1}{2}$

RHS = $(1+1) - H_{1+1} = 2 - H_2 = 2 - \frac{3}{2} = \frac{1}{2}$

Thus, the equation holds for n=1.

Inductive hypothesis: Assume for an arbitrary n=k, that

$$\sum_{i=1}^{k} \frac{i}{i+1} = (k+1) - H_{k+1}$$

Inductive step: Prove for n=k+1 that

$$\sum_{i=1}^{k+1} \frac{i}{i+1} = (k+2) - H_{k+2}$$

$\sum_{i=1}^{k+1} \frac{i}{i+1} = (\sum_{i=1}^{k} \frac{i}{i+1}) + \frac{k+1}{k+2}$, splitting the sum

$= (k+1) - H_{k+1} + \frac{k+1}{k+2}$, using the inductive hypothesis

$= (k+1) - H_{k+1} + \frac{k+2-1}{k+2}$, rewriting the numerator

$= (k+1) - H_{k+1} + 1 - \frac{1}{k+2}$, splitting the fraction

$= (k+2) - (H_{k+1} + \frac{1}{k+2})$, combining terms, factoring out a minus sign

$= (k+2) - H_{k+2}$, using the definition of a Harmonic number

This completes the proof of the inductive step. We can conclude for all positive integers n that

$$\sum_{i=1}^{n} \frac{i}{i+1} = (n+1) - H_{n+1}$$

3) Use induction on n to prove the following inequality for all positive integers n:

$$\sum_{i=0}^{n} 3^i < \frac{3^{n+1}}{2}$$

Solution

Base case: n = 1, LHS = $\sum_{i=0}^{n} 3^i = 3^0 + 3^1 = 4$, RHS = $\frac{3^2}{2} = \frac{9}{2}. \frac{9}{2} > 4$, so the base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=0}^{k} 3^i < \frac{3^{k+1}}{2}$$

Inductive Step: Prove for n = k+1 that

$$\sum_{i=0}^{k+1} 3^i < \frac{3^{k+2}}{2}$$

$\sum_{i=0}^{k+1} 3^i = \left(\sum_{i=0}^{k} 3^i\right) + 3^{k+1}$

$< \frac{3^{k+1}}{2} + 3^{k+1}$

$= 3^{k+1}(\frac{1}{2} + 1)$

$= 3^{k+1}(\frac{3}{2})$

$= \frac{3^{k+2}}{2}$

4) The Fibonacci numbers are defined as follows: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, for all integers $n>1$. Prove the following formula for all positive integers n:

$$\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}$$

Solution

Base case: n = 1, LHS = $\sum_{i=1}^{1} \frac{F_{i-1}}{2^i} = \frac{F_0}{2^1} = 0$, RHS = $1 - \frac{F_{1+2}}{2^1} = 1 - \frac{2}{2} = 0$

Thus the given assertion is true for n = 1.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+2}}{2^k}$$

Inductive Step: Prove for n = k+1 that

$$\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+3}}{2^{k+1}}$$

$\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = \left(\sum_{i=1}^{k} \frac{F_{i-1}}{2^i}\right) + \frac{F_k}{2^{k+1}}$, splitting the sum

$= 1 - \frac{F_{k+2}}{2^k} + \frac{F_k}{2^{k+1}}$, using the inductive hypothesis

$= 1 - \frac{2F_{k+2}}{2^{k+1}} + \frac{F_k}{2^{k+1}}$, getting a common denominator and using an exponent rule

$= 1 - (\frac{2F_{k+2} - F_k}{2^{k+1}})$, factoring out a minus sign

$= 1 - (\frac{F_{k+2} + F_{k+2} - F_k}{2^{k+1}})$, splitting $2F_{k+2}$

$= 1 - (\frac{F_{k+2} + F_{k+1}}{2^{k+1}})$, using the definition of Fibonacci numbers

$= 1 - (\frac{F_{k+3}}{2^{k+1}})$, using the definition of Fibonacci numbers

This completes the proof of the inductive step. It follows that for all positive integers n

$$\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}$$

5) Using induction on n, prove for all non-negative integers n that $9 \mid (2^{2n} + 6n - 1)$.

Solution

Base case: $n = 0$, $2^{2(0)} + 6(0) – 1 = 2^0 + 0 – 1 = 1 – 1 = 0$. $9 \mid 0$ because $0 = 0 \times 9$.
Thus the given assertion is true for $n = 0$.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer $n = k$ that

$$9 \mid (2^{2k} + 6k - 1).$$

This means there exists an integer c such that $2^{2k} + 6k – 1 = 9c$, or $2^{2k} = 9c – 6k + 1$.

Inductive step: Prove for $n = k+1$ that

$$9 \mid (2^{2(k+1)} + 6(k+1) - 1).$$

$$\begin{aligned}
2^{2(k+1)} + 6(k+1) – 1 &= 2^{2k+2} + 6k + 6 – 1 \\
&= 2^2 2^{2k} + 6k + 5 \\
&= 4(2^{2k}) + 6k + 5 \\
&= 4(9c – 6k + 1) + 6k + 5 \\
&= 36c – 24k + 4 + 6k + 5 \\
&= 36c – 18k + 9 \\
&= 9(4c – 2k + 1)
\end{aligned}$$

Since c and k are integers, it follows that $4c – 2k + 1$ is also an integer. Thus, we've proven that $9 \mid (2^{2(k+1)} + 6(k+1) - 1)$, completing the proof of the inductive step.

It follows that for all non-negative integers n, $9 \mid (2^{2n} + 6n - 1)$.

6) Define the sequence t_n as follows: $t_0 = 7$, $t_1 = 10$, $t_n = 3t_{n-1} - 2t_{n-2}$, for all integers $n \geq 2$. Prove, using strong induction on n with 2 base cases, that for all non-negative integers n,

$$t_n = 4 + 3(2^n).$$

Solution

Base cases: $n = 0$, LHS = $t_0 = 7$, RHS = $4 + 3(2^0) = 4 + 3(1) = 7$

$n = 1$, LHS = $t_1 = 10$, RHS = $4 + 3(2^1) = 4 + 3(2) = 10$

Thus, the formula holds for both n=0 and n=1, proving the base cases.

Inductive Hypothesis: Assume for all non-negative integers n, $0 \leq n \leq k$, where k is an arbitrarily chosen positive integer, that $t_n = 4 + 3(2^n)$.

Inductive Step: Prove for $n = k+1$ that $t_{k+1} = 4 + 3(2^{k+1})$.

$t_{k+1} = 3t_k - 2t_{k-1}$, using the definition of the sequence t_n.

$= 3(4 + 3(2^k)) - 2(4 + 3(2^{k-1}))$, using the inductive hypothesis for both k and $k - 1$. Since $k+1 \geq 2$, it follows that $k - 1 \geq 0$ so the inductive hypothesis applies to both cases.

$= 12 + 9(2^k) - 8 - 6(2^{k-1})$, multiplying out terms

$= 4 + 9(2^k) - 3(2)(2^{k-1})$, rewriting 6

$= 4 + 9(2^k) - 3(2^k)$, using the exponent rule

$= 4 + 6(2^k)$, combining like terms

$= 4 + 3(2)(2^k)$, rewriting 6

$= 4 + 3(2^{k+1})$, exponent rule

This completes the proof of the inductive step. It follows that for all non-negative integers, n,

$t_n = 4 + 3(2^n)$.

7) Prove, for all positive integers n, using induction on n that

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^n = \frac{1}{4}\begin{bmatrix} 5^n+3 & 1-5^n \\ 3(1-5^n) & 3(5^n)+1 \end{bmatrix}.$$

Solution

Base case: n = 1, LHS = $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^1 = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

RHS = $\frac{1}{4}\begin{bmatrix} 5^1+3 & 1-5^1 \\ 3(1-5^1) & 3(5^1)+1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 & -4 \\ -12 & 16 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

Thus, the given formula holds for n = 1.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^k = \frac{1}{4}\begin{bmatrix} 5^k+3 & 1-5^k \\ 3(1-5^k) & 3(5^k)+1 \end{bmatrix}$$

Inductive Step: Prove for n = k+1 that

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^{k+1} = \frac{1}{4}\begin{bmatrix} 5^{k+1}+3 & 1-5^{k+1} \\ 3(1-5^k) & 3(5^k)+1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^{k+1} = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^k \times \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

$= \frac{1}{4}\begin{bmatrix} 5^k+3 & 1-5^k \\ 3(1-5^k) & 3(5^k)+1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$, using the inductive hypothesis

$= \frac{1}{4}\begin{bmatrix} 2(5^k+3)-3(1-5^k) & -(5^k+3)+4(1-5^k) \\ 3(1-5^k)2-3(3(5^k)+1) & -3(1-5^k)+4(3(5^k)+1) \end{bmatrix}$, matrix mult

$= \frac{1}{4}\begin{bmatrix} 2(5^k)+6-3+3(5^k) & -5^k-3+4-4(5^k) \\ 6-6(5^k)-9(5^k)-3 & -3+3(5^k)+12(5^k)+4 \end{bmatrix}$, algebra

$= \frac{1}{4}\begin{bmatrix} 5(5^k)+3 & 1-5(5^k) \\ 3-15(5^k) & 15(5^k)+1 \end{bmatrix}$, algebra

$= \frac{1}{4}\begin{bmatrix} (5^{k+1})+3 & 1-(5^{k+1}) \\ 3-3(5)(5^k) & 3(5)(5^k)+1 \end{bmatrix}$, algebra

$$= \frac{1}{4}\begin{bmatrix} (5^{k+1})+3 & 1-(5^{k+1}) \\ 3(1-(5^{k+1})) & 3(5^{k+1})+1 \end{bmatrix}$$

This completes the proof of the inductive step. We can conclude that for all positive integers n,

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^n = \frac{1}{4}\begin{bmatrix} 5^n + 3 & 1 - 5^n \\ 3(1 - 5^n) & 3(5^n) + 1 \end{bmatrix}.$$