COT 3100 Recitation Solutions: Induction Prep (Sums, Matrices, Rec. Rel)

1) An arithmetic sequence $a_1, a_2, a_3, \ldots$ is such that $a_{10} = 40$ and $a_{25} = 10$. Determine both a_1 and the sum of the first 50 terms of the sequence.

Solution

Let d be the common difference for the sequence.

$a_{25} - a_{10} = 15d$
$10 - 40 = 15d$
$-30 = 15d$
$d = -2$

$a_{10} = a_1 + 9d$
$40 = a_1 + 9(-2)$
$a_1 = 58$
$a_{50} = a_1 + 49d = 58 + 49(-2) = -40$

$$S_{50} = \frac{50(a_1 + a_{50})}{2} = 25(58 - 40) = 25 \times 18 = \mathbf{450}$$

2) An infinite geometric sequence $a_1, a_2, a_3, \ldots$ has a sum of 10. If the odd indexed terms (a_1, a_3, etc.) have a sum of 7, what is the common ratio of the sequence? What is the first term of the sequence?

Solution

Let the common ratio of the sequence equal r.

Using the first piece of information, we have: $\frac{a_1}{1-r} = 10$.

Note that the sequence $a_1, a_3, a_5, \ldots$ is just an infinite geometric sequence with common ratio r^2. Thus, we have: $\frac{a_1}{1-r^2} = 7$.

There are two typical ways to proceed from here. Perhaps the most obvious is to solve the system of equations. First solve for a_1 in the first equation: $a_1 = 10(1-r)$, and then substitute this into the second equation: $\frac{10(1-r)}{1-r^2} = 7$. The denominator factors to help us:

$\frac{10(1-r)}{(1-r)(1+r)} = 7$

$\frac{10}{(1+r)} = 7$

$10 = 7(1+r)$

$10 = 7 + 7r$

$7r = 3$

$r = \frac{3}{7}$

Now backsubstitute:

$\frac{a_1}{1-\frac{3}{7}} = 10$

$\frac{a_1}{\frac{4}{7}} = 10$

$\boldsymbol{a_1 = \frac{40}{7}}$

The other solution path is to realize that the sum of a_2, a_4, a_6, ... is 3, but that this sum is also the sum of a_1r, a_3r, a_5r, ..., which is equal to $r(a_1 + a_3 + a_5 + ...) = 7$, thus $r = \frac{3}{7}$. From there back substitute like the previous solution.

3) Determine the following summation in terms of n: $\sum_{i=n+1}^{2n}(3i-2)$.

Solution

One solution is to realize that his summation is an arithmetic series with n terms. The first term is $3(n+1) - 2 = 3n+1$ and the last term is $3(2n) - 2 = 6n - 2$. The corresponding sum is $\frac{n(3n+1+6n-2)}{2}$ $= \boldsymbol{\frac{n(9n-1)}{2}}$.

Alternatively, use the standard summation rules taught:

$$\begin{aligned}
\sum_{i=n+1}^{2n}(3i-2) &= \left(\left(\sum_{i=1}^{2n} 3i\right) - \left(\sum_{i=1}^{n} 3i\right)\right) - \left(\sum_{i=n+1}^{2n} 2\right) \\
&= \frac{3(2n)(2n+1)}{2} - \frac{3n(n+1)}{2} - 2((2n) - (n+1) + 1) \\
&= 3n(2n+1) - \frac{3n(n+1)}{2} - 2(n) \\
&= 6n^2 + 3n - \frac{3}{2}n^2 - \frac{3}{2}n - 2n \\
&= \frac{9}{2}n^2 - \frac{1}{2}n \\
&= \boldsymbol{\frac{n(9n-1)}{2}}
\end{aligned}$$

4) Determine the following summation in terms of n: $\sum_{i=1}^{2n}(3i^2 - 4i)$.

Solution

$$\sum_{i=1}^{2n}(3i^2 - 4i) = \left(\sum_{i=1}^{2n} 3i^2\right) - \left(\sum_{i=1}^{2n} 4i\right)$$

$$= \frac{3(2n)(2n+1)(4n+1)}{6} - \frac{4(2n)(2n+1)}{2}$$

$$= n(2n+1)(4n+1) - 4n(2n+1)$$

$$= n(2n+1)[(4n+1) - 4]$$

$$= n(2n+1)(4n-3)$$

5) Determine the following infinite summation: $\sum_{i=1}^{\infty}(2i-1)(\frac{1}{2})^i$.

Solution

Let S equal the summation shown above:

$$S = 1 \times (\frac{1}{2})^1 + 3 \times (\frac{1}{2})^2 + 5 \times (\frac{1}{2})^3 + \cdots$$

Multiply S through by $\frac{1}{2}$ and rewrite the equation:

$$\frac{S}{2} = \qquad 1 \times (\frac{1}{2})^2 + 3 \times (\frac{1}{2})^3 + 5 \times (\frac{1}{2})^4 + \cdots$$

Subtracting the bottom equation from the top, we get:

$$\frac{S}{2} = 1 \times (\frac{1}{2})^1 + 2 \times (\frac{1}{2})^2 + 2 \times (\frac{1}{2})^3 + \cdots$$

Multiply through by 2, then solve for the infinite geometric series starting at the second term to complete the solution.

$$S = 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 1 + \frac{1}{1 - \frac{1}{2}} = \mathbf{3}$$

6) Determine the following matrix sum: $\begin{bmatrix} 6 & 3 & -2 \\ 5 & -9 & 7 \end{bmatrix} + \begin{bmatrix} -2 & 12 & 3 \\ 6 & 4 & 8 \end{bmatrix}$.

<u>Solution</u>

$$\begin{bmatrix} 6 & 3 & -2 \\ 5 & -9 & 7 \end{bmatrix} + \begin{bmatrix} -2 & 12 & 3 \\ 6 & 4 & 8 \end{bmatrix} = \begin{bmatrix} \mathbf{4} & \mathbf{15} & \mathbf{1} \\ \mathbf{11} & \mathbf{-5} & \mathbf{15} \end{bmatrix}$$

7) Determine the following matrix product, $\begin{bmatrix} 2^n & 1-2^n \\ 3 & 2^{n-1} \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$, in terms of n.

<u>Solution</u>

$$\begin{bmatrix} 2^n & 1-2^n \\ 3 & 2^{n-1} \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3(2^n)+1(1-2^n) & 2(2^n)-(1-2^n) \\ 3(3)+1(2^{n-1}) & 3(2)-1(2^{n-1}) \end{bmatrix}$$

$$= \begin{bmatrix} 3(2^n)+1-2^n & 2(2^n)-1+2^n \\ 2^{n-1}+9 & -2^{n-1}+6 \end{bmatrix}$$

$$= \begin{bmatrix} 2(2^n)+1 & 3(2^n)-1 \\ 2^{n-1}+9 & -2^{n-1}+6 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{2^{n+1}+1} & \mathbf{3(2^n)-1} \\ \mathbf{2^{n-1}+9} & \mathbf{-2^{n-1}+6} \end{bmatrix}$$

8) Let the Fibonacci Sequence be defined as follows: $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$, for all integers n ≥ 2. Determine F_{n+2} in terms of F_{n-1} and F_{n-2}.

<u>Solution</u>

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n \\ &= (F_n + F_{n-1}) + F_n \\ &= 2F_n + F_{n-1} \\ &= 2(F_{n-1} + F_{n-2}) + F_{n-1} \\ &= 2F_{n-1} + 2F_{n-2} + F_{n-1} \\ &= \mathbf{3F_{n-1} + 2F_{n-2}} \end{aligned}$$

9) Once again, let F_n denote the n^{th} Fibonacci number (defined in the previous question). Determine and simplify the following matrix product, in terms of n:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} & F_{n-1} + F_{n-2} \\ F_n - F_{n-1} & F_{n-1} - F_{n-2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F_{n+1}} & \boldsymbol{F_n} \\ \boldsymbol{F_{n-2}} & \boldsymbol{F_{n-3}} \end{bmatrix}$$

10) Let a, b, x and y be arbitrary positive constants. Let $f(n) = ax^n + by^n$ for all non-negative integers, n.

For an arbitrary positive integer, n, prove that

$f(n+1) = (x+y)f(n) - xyf(n-1)$.

Solution

This is easier to prove starting with the right-hand side:

$$\begin{aligned} (x+y)f(n) - xyf(n-1) &= (x+y)(ax^n + by^n) - xy(ax^{n-1} + by^{n-1}) \\ &= ax^{n+1} + bxy^n + ayx^n + by^{n+1} - ayx^n - bxy^n \\ &= ax^{n+1} + by^{n+1} \\ &= f(n+1) \end{aligned}$$